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A Memoir on the Abelian and Theta Functions.

By Professor Cayley.

CONTINUED FROM VOL. V. (1882), pp. 137-179.

Chapter IV. The Major Function $(x, y, z)^{n-2}_{12}$, Continued.

The Conversion, Fixed Curve a Quartic, Continued. Art. Nos. 74 to 82.

74. I resume the question considered *ante* Nos. 66 to 73. The general problem where the fixed curve is any given curve whatever, has recently been solved in a very complete and elegant form by Dr. Nöther, in the two notes "Zur Reduction algebraischer Differentialausdrücke auf der Normalförmen" and "Ueber die algebraischen Differentialausdrücke, 2$^{\text{e}}$ Note," Sitzungsb. der phys-med. Soc. zu Erlangen, 10 Dec. 1883 and 14 Jan. 1884. I consider here the case of the quartic curve, $n=4$, and connect his result with my former investigations.

We have the differential

$$\frac{(x, y, z)^{n-2}_{12}\,d\omega}{012},\ =\frac{\Omega_{12}\,d\omega}{012}$$

where Ω_{12}, or as I also write it $\Omega\,(0;\,1,\,2;\,3,\,4,\,5)$, is a rational and integral function of the degree $(n-2=)\ 2$ in the current coordinates (x, y, z): it depends also on the parametric points 1, 2, which are points on the quartic, coordinates (x_1, y_1, z_1), (x_2, y_2, z_2) respectively; and on $(p=)\ 3$ other points 3, 4, 5 on the quartic, coordinates (x_3, y_3, z_3), (x_4, y_4, z_4), (x_5, y_5, z_5) respectively. The curve $\Omega=0$ is a conic which is taken to pass through the dps (none in the present case) and through the $(n-2=)\ 2$ residues of the parametric points, and the function Ω is such that on writing therein (x_1, y_1, z_1) for (x, y, z) it becomes $=(n.1^{n-1}2$* $=)$ 4.1^32 (viz. we have $\Omega\,(1;\,1,\,2;\,3,\,4,\,5)=4.1^32$, which implies also $\Omega\,(2;\,1,\,2;\,3,\,4,\,5)$

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*$\left(x_2\frac{d}{dx_1}+y_2\frac{d}{dy_1}+z_2\frac{d}{dz_1}\right)f(x_1, y_1, z_1)=n.1^{n-1}2$. See No. 2.

$= 4.12^3$): so defined, the function would contain ($p =$) 3 arbitrary constants, but these are determined so that the curve $\Omega = 0$ passes through the 3 points 3, 4, 5 on the quartic: and the function Ω, $= \Omega(0\colon 1,\ 2\colon 3,\ 4,\ 5)$ is thus a completely determinate function, rational and integral of the degree 2 in the coordinates (x, y, z) of the current point, and rational in the coordinates of the other five points respectively. I call to mind that 012 denotes the determinant formed with the coordinates (x, y, z), etc., of the points 0, 1, 2 respectively: the like notation is used throughout.

75. The function $\Omega(0;\ 1,\ 2 : 3,\ 4,\ 5)$ is in fact the function Ω' of No. 43 with only the further condition in regard to the points 3, 4, 5 of the quartic; viz., Ω is the function determined by the equation

$$\left|\begin{array}{ll} (x\ ,\ y\ ,\ z\ )^2 & ,\ \Omega \\ 1\,(x_1,\ y_1,\ z_1)^2 & ,\ 4.1^3 2 \\ 2\,(x_1,\ y_1,\ z_1)(x_2,\ y_2,\ z_2), & 6.1^2 2^2 \\ 1\,(x_2,\ y_2,\ z_2)^2 & ,\ 4.12^3 \\ \quad (x_3,\ y_3,\ z_3)^2 & ,\quad 0 \\ \quad (x_4,\ y_4,\ z_4)^2 & ,\quad 0 \\ \quad (x_5,\ y_5,\ z_5)^2 & ,\quad 0 \end{array}\right| = 0:$$

this is of the form $\quad M\Omega + \square = 0,$

and as appears in No. 46, M is $= 123.124.125.345$. Hence writing $\square(0;\ 1,\ 2;\ 3,\ 4,\ 5)$ for $\square$, we have

$$\Omega, = \Omega(0;\ 1,\ 2;\ 3,\ 4,\ 5), = \frac{-\square(0; 1, 2; 3, 4, 5)}{123.124.125.345}.$$

Hence further writing Q, $= Q(0;\ 1,\ 2;\ 3,\ 4,\ 5)$, $= \dfrac{\Omega(0; 1, 2; 3, 4, 5)}{012}$, so that the differential is $Qd\omega$, $= Q(0;\ 1,\ 2;\ 3,\ 4,\ 5)\,d\omega$, we have

$$Q = Q(0;\ 1,\ 2; 3,\ 4,\ 5), = \frac{-\square(0; 1, 2; 3, 4, 5)}{012.123.124.125.345},$$

which is of the form $\quad Q = \dfrac{0^2\,\overline{12}^4\,\overline{345}^2}{0^1\,\overline{12}^4\,\overline{345}^2}, = 0^1\,\overline{12345}^0,$

viz., Q is a rational fraction where the numerator is of the degree 2, and the denominator of the degree 1 as regards the coordinates (x, y, z) of the current point: but the numerator and denominator are each of the degree 4 as regards the coordinates of the points 1, 2 separately, and of the degree 2 as regards the coordinates of the points 3, 4, 5 separately: that is Q is of the degree 1 as regards the coordinates (x, y, z), but of the degree 0 as regards the coordinates of the points 1, 2, 3, 4, 5 separately.

76. The signification of the symbol of quasi-differentiation ∂ (applicable only to a function of the degree 0 in the coordinates to which the differentiations have reference) is explained *ante* No. 60, the function Q just mentioned is of the degree 0 in regard to the coordinates of each of the points 1, 2, 3, 4, 5; and it can thus be operated upon by the symbols ∂_1, ∂_2, ∂_3, ∂_4, ∂_5 respectively. Observe in particular that we have $\partial_1 Q(0;\, 1,\, 2;\, 3,\, 4,\, 5) = \overline{01}^1\,\overline{2345}^0$, viz., it is of the degree 1 in the coordinates of the points 0 and 1 respectively, but of the degree 0 in regard to the coordinates of the points 2, 3, 4, 5 respectively.

77. This being so we may consider the function

$$\begin{aligned} H(0;\, 1,\, 2;\, 3,\, 4,\, 5;\, 6,\, 7,\, 8) = {} & \partial_1 Q(0;\, 1,\, 2;\, 3,\, 4,\, 5) \\ & + \partial_3 Q(1;\, 3,\, 2;\, 6,\, 4,\, 5).\frac{045}{345} \\ & + \partial_4 Q(1;\, 4,\, 2;\, 3,\, 7,\, 5).\frac{053}{453} \\ & + \partial_5 Q(1;\, 5,\, 2;\, 3,\, 4,\, 8).\frac{034}{534}, \end{aligned}$$

where 6, 7, 8 are arbitrary points on the quartic; the functions

$$\partial_3 Q(1;\, 3,\, 2;\, 6,\, 4,\, 5),\quad \partial_4 Q(1;\, 4,\, 2;\, 3,\, 7,\, 5),\quad \partial_5 Q(1;\, 5,\, 2;\, 3,\, 4,\, 8)$$

are functions of the same form as $\partial_1 Q(0;\, 1,\, 2;\, 3,\, 4,\, 5)$, and derived from it by changing in each case the current point 0 into the parametric point 1, and by further changing in the three cases this parametric point into the points 3, 4, 5 respectively, and replacing the corresponding point 3, 4 or 5 by the new arbitrary point 6, 7 or 8. Further 045, etc., denote determinants as above; so that in H each of the last three terms is in fact as regards the point 0 a mere linear function of the coordinates (x, y, z) of this point.

We have $\partial_3 Q(1;\, 3,\, 2;\, 6,\, 4,\, 5) = \overline{13}^1\,\overline{2645}^0$, and hence this function into $\frac{045}{345}$ is $= \overline{01}^1\,\overline{23456}^0$; and so for the third and fourth terms of H: thus each of the four terms of H is $= \overline{01}^1\,\overline{2345678}^0$, of the degree 1 in the coordinates of the points 0 and 1 respectively, but of the degree 0 in the coordinates of the other points 2, 3, 4, 5, 6, 7, 8 respectively.

Nöther's conversion-theorem consists herein, that the function

$$H(0;\, 1,\, 2;\, 3,\, 4,\, 5;\, 6,\, 7,\, 8)$$

is unaltered by the interchange of the two points 0, 1; or putting for shortness

$$H(0;\, 1,\, 2;\, 3,\, 4,\, 5;\, 6,\, 7,\, 8) = H_1(0),$$

the theorem is

$$H_1(0) = H_0(1).$$

78. We have No. 59,

$$\frac{\frac{1}{2}\Omega_{12}}{012}=\frac{-01^3.02^3+01^22.012^2+0^212.1^22^2}{012.1^22^2},\ =\{0^2 12\},$$

or as for greater simplicity I write it $= 0^212$,

viz. 0^212 is now written instead of $\{0^212\}$ to denote the function just given as the value of $\frac{\frac{1}{2}\Omega_{12}}{012}$: Ω_{12} is thus $= 2.012.0^212$, viz. this is a particular form of Ω_{12} satisfying the conditions that $\Omega_{12}=0$ is a conic passing through the residues of the points 1, 2, and such that Ω_{12} on writing therein (x_1, y_1, z_1) for (x, y, z) becomes $= 4.1^32$: hence the general form of the function satisfying these conditions is $= 2.012\,\{0^212 +$ arbitrary linear function of $(x, y, z)\}$. The before-mentioned function $\Omega(0; 1, 2; 3, 4, 5)$ is a function satisfying these conditions and the further conditions that the conic $\Omega = 0$ shall pass through the three points 3, 4, 5 on the quartic: these further conditions serve to determine the linear function: and we at once obtain

$$\frac{\frac{1}{2}\Omega(0;1,2;3,4,5)}{012}=0^212-3^212\,\frac{045}{345}-4^212\,\frac{053}{453}-5^212\cdot\frac{034}{534},$$

viz. the value of Ω given by this equation, on writing therein $0 = 3$, 4 or 5, becomes as it should do $= 0$.

79. We thus have

$$Q(0;1,2;3,4,5)=0^212-3^212\,\frac{045}{345}-4^212\,\frac{053}{453}-5^212\,\frac{034}{534},$$

and Nöther's conversion-equation becomes

$$\begin{aligned}
&\partial_1\left\{0^212-3^212\,\frac{045}{345}-4^212\,\frac{053}{453}-5^212\,\frac{034}{534}\right\}\\
&+\partial_3\left\{1^232-6^232\,\frac{145}{645}-4^232\,\frac{156}{456}-5^232\,\frac{164}{564}\right\}\cdot\frac{045}{345}\\
&+\partial_4\left\{1^242-3^242\,\frac{175}{375}-7^242\,\frac{153}{753}-5^242\,\frac{137}{537}\right\}\cdot\frac{053}{453}\\
&+\partial_5\left\{1^252-3^252\,\frac{148}{348}-4^252\,\frac{183}{483}-5^248\,\frac{134}{834}\right\}\cdot\frac{034}{534}\\
=\ &\partial_0\left\{1^202-3^202\,\frac{145}{345}-4^202\,\frac{153}{453}-5^202\,\frac{134}{534}\right\}\\
&+\partial_3\left\{0^232-6^232\,\frac{045}{645}-4^232\,\frac{056}{456}-5^232\,\frac{064}{564}\right\}\cdot\frac{145}{345}\\
&+\partial_4\left\{0^242-3^242\,\frac{075}{375}-7^242\,\frac{053}{753}-5^242\,\frac{037}{537}\right\}\cdot\frac{153}{453}\\
&+\partial_5\left\{0^252-3^252\,\frac{048}{348}-4^252\,\frac{083}{483}-5^248\,\frac{034}{834}\right\}\cdot\frac{134}{534}
\end{aligned}$$

an equation where the functions operated on with the ∂'s are only functions such as $0^2 12$; for there is not any determinant operated upon which contains the number which is the suffix of the ∂ which operates upon it.

80. Taking all the terms over to the left-hand side there are in all 32 terms: but of these 3 + 3 destroy each other, and 6 + 6 unite in pairs into 6 terms: there are thus in all 7 + 7 + 6, = 20 terms: viz. multiplying the whole equation by 345, it is found that the equation becomes

$$\begin{array}{lll}
 & \text{or as this may be written} & \text{where} \\
345(\quad \partial_1 0^2 12 - \partial_0 1^2 02) & 345\,\boxed{012} & \boxed{012} = \partial_1 0^2 12 - \partial_0 1^2 02, \text{ etc.,} \\
-045(-\partial_3 1^2 32 + \partial_1 3^2 12) & -045\,\boxed{312} & \\
-053(-\partial_4 1^2 42 + \partial_1 4^2 12) & -305\,\boxed{412} & \\
-034(-\partial_5 1^2 52 + \partial_1 5^2 12) & -340\,\boxed{512} & \\
-145(\quad \partial_3 0^2 32 - \partial_0 3^2 02) & -145\,\boxed{032} & \\
-153(\quad \partial_4 0^2 42 - \partial_0 4^2 02) & -315\,\boxed{042} & \\
-134(\quad \partial_5 0^2 52 - \partial_0 5^2 02) & -341\,\boxed{052} & \\
+013(-\partial_4 5^2 42 + \partial_5 4^2 52) & +301\,\boxed{452} & \\
+014(-\partial_5 3^2 52 + \partial_3 5^2 32) & +140\,\boxed{532} & \\
+015(-\partial_3 4^2 32 + \partial_4 3^2 42) & +015\,\boxed{342} & \\
=0, & =0, &
\end{array}$$

viz. the equation is

$$\Sigma \pm 345\,\boxed{012} = 0,$$

the nine terms which follow the first term $345\,\boxed{012}$ of the sum being obtained by the interchanges of 0, 1 (one or each) with the 3, 4, 5, each interchange giving rise to a sign —.

81. In obtaining the foregoing result, we have, for instance, a pair of terms

$$\partial_4\, 5^2 42 \frac{-137.053 + 037.153}{537},\ = \partial_4\, 5^2 42 \frac{-013.537}{537},\ = \partial_4\, 5^2 42 (-013),$$

viz. this depends on the equation

$$137.053 - 037.153 - 537.013 = 0,$$

or say

$$-137.035 + 037.135 + 013.357 = 0,$$

an identity which, in a form which will be readily understood, may be written

$$\text{det.} \begin{vmatrix} 0137 \\ 013735 \end{vmatrix} = 0.$$

Similarly the two terms which contain $\partial_5\, 4^2 52$ combine into the single term $\partial_5\, 4^2 52(013)$: and the two new terms taken together are

$$013(-\partial_4\, 5^2 42 + \partial_5\, 4^2 52).\ = 301\ \boxed{452}\,.$$

82. The proof of the identity,

$$\Sigma \pm 345\ \boxed{012} = 0,$$

depends on the property of the function

$$\boxed{012},\ = \partial_1\, 0^2 12 - \partial_0\, 1^2 02,$$

enuntiated No. 67, and proved *à posteriori* by the tedious calculation Nos. 70 to 73, viz. in No. 67, writing 2 in place of 3, this is:—$\partial_1\, 0^2 12 - \partial_0\, 1^2 02$ is equal to the difference of two functions, the first of them linear in the coordinates (x, y, z) of the point 0, but depending also on the coordinates of the points 1 and 2; the second of them linear in the coordinates (x_1, y_1, z_1) of the point 1 but depending also on the coordinates of the points 0 and 2. Or what is the same thing the property is

$$\boxed{012} = A_{12}x + B_{12}y + C_{12}z - (A_{02}x_1 + B_{02}y_1 + C_{02}z_1),$$

where A_{12}, B_{12}, C_{12} are functions of (x_1, y_1, z_1), (x_2, y_2, z_2), and A_{02}, B_{02}, C_{02} are the like functions of (x, y, z), (x_2, y_2, z_2).

Substituting such values in the sum $\Sigma \pm 345\ \boxed{012}$, but writing down only the terms which contain x, these are

$$\begin{array}{lll} & 345\,(A_{12}x - A_{02}x_1) & \\ -\,045\,(A_{12}x_3 - A_{32}x_1) & -\,145\,(A_{32}x - A_{02}x_3) & +\,301\,(A_{52}x_4 - A_{42}x_5) \\ -\,305\,(A_{12}x_4 - A_{42}x_1) & -\,315\,(A_{42}x - A_{02}x_4) & +\,140\,(A_{32}x_5 - A_{52}x_3) \\ -\,340\,(A_{12}x_5 - A_{52}x_1) & -\,341\,(A_{52}x - A_{02}x_5) & +\,015\,(A_{42}x_3 - A_{32}x_4). \end{array}$$

This is

$$\begin{aligned} = \quad & A_{02}\,(-x_1 345 + x_3 145 + x_4 315 + x_5 341) \\ & + A_{12}\,(\quad x\,345 - x_3 045 - x_4 305 - x_5 340) \\ & + A_{32}\,(\quad x_1 045 - x\,145 + x_5 140 - x_4 015) \\ & + A_{42}\,(-x\,315 + x_3 015 + x_1 305 - x_5 301) \\ & + A_{52}\,(-x\,341 + x_4 301 + x_1 340 - x_3 140) \end{aligned}$$

where the coefficient of each of the A's is identically $=0$: and similarly the terms in y and the terms in z are each $=0$. We have thus the proof of the identity

$$\Sigma \pm 345\,\boxed{012} = 0,$$

that is, of the conversion-equation $H_1(0) = H_0(1)$.

The Syzygy—Fixed Curve a Quartic. Art. No. 83.

I revert to the theory of the Syzygy, *ante* No. 59.

83. We have

$$Q(0;\,1,\,2;\,3,\,4,\,5) = 0^2 12 - 3^2 12\,\frac{045}{345} - 4^2 12\,\frac{053}{453} - 5^2 12\,\frac{034}{534},$$

or if for convenience we take instead of 1, 2, the parametric points to be α, β coordinates $(x_\alpha,\,y_\alpha,\,z_\alpha)$ and $(x_\beta,\,y_\beta,\,z_\beta)$ respectively, then this equation is

$$Q_{\alpha\beta} = Q\,(0;\,\alpha,\,\beta;\,3,\,4,\,5) = 0^2\alpha\beta - 3^2\alpha\beta\,.\frac{045}{345} - 4^2\alpha\beta\,\frac{053}{453} - 5^2\alpha\beta\,\frac{034}{534}\,.$$

Considering a new parametric point γ, and forming the like functions $Q_{\beta\gamma}$ and $Q_{\gamma\alpha}$, it is to be shown that we have identically

$$Q_{\alpha\beta} + Q_{\beta\gamma} + Q_{\gamma\alpha} = 0\,.$$

To prove this observe that in the equation at the end of No. 59, Δ, ρ, σ, τ denote 123, 023, 031, 012 respectively. Hence writing therein α, β, γ in place of 1, 2, 3 respectively, and putting A, B, C for the coefficients (including therein the factor $\frac{1}{\Delta^2}$) of ρ, σ, τ respectively, the equation is

$$0^2\beta\gamma + 0^2\gamma\alpha + 0^2\alpha\beta = A\,.\,0\beta\gamma + B\,.\,0\gamma\alpha + C\,.\,0\alpha\beta,$$

where A, B, C are absolute constants (functions, that is, of the coefficients of the quartic) each divided by $(\alpha\beta\gamma)^2$. We hence obtain

$$\begin{aligned} (Q_{\beta\gamma} + Q_{\gamma\alpha} + Q_{\alpha\beta})\,.\,345 = \quad & 345\,(A\,.\,0\beta\gamma + B\,.\,0\gamma\alpha + C\,.\,0\alpha\beta) \\ & - 045\,(A\,.\,3\beta\gamma + B\,.\,3\gamma\alpha + C\,.\,3\alpha\beta) \\ & - 053\,(A\,.\,4\beta\gamma + B\,.\,4\gamma\alpha + C\,.\,4\alpha\beta) \\ & - 034\,(A\,.\,5\beta\gamma + B\,.\,5\gamma\alpha + C\,.\,5\alpha\beta) \end{aligned}$$

and on the left-hand side the whole coefficient of A is $=0$; viz. this coefficient has the value det. $\begin{vmatrix} 0345 \\ 0345\beta\gamma \end{vmatrix}$, which is $=0$. Similarly the whole coefficient of B is $=0$, and the whole coefficient of C is $=0$: and we have thus the required result

$$Q_{\beta\gamma}+Q_{\gamma\alpha}+Q_{\alpha\beta}=0.$$

The syzygy is thus obtained in a more perfect form than in No. 59; viz. by considering (instead of $0^2\alpha\beta$) the new form $Q_{\alpha\beta}$, then instead of a sum which is a linear function of the coordinates (x, y, z) we obtain a sum $=0$.

The Fixed Curve a Cubic—Syzygy and Conversion. Art. Nos. 84, 85.

84. In the case fixed curve a cubic (see Nos. 58 and 64) the analogous formulæ are

$$\tfrac{1}{3}\Omega(0;\,1,\,2;\,3)=\frac{1^22.023+12^2.031}{123},\ =\widetilde{012}-\frac{\widetilde{123}}{123}012 \text{ (see No. 49)},$$

that is

$$Q(0;\,1,\,2;\,3),\ =\frac{\frac{1}{3}\Omega}{012},\ =\frac{\widetilde{012}}{012}-\frac{\widetilde{123}}{123},\ =\{012\}-\{123\},$$

where 1, 2 are the parametric points: 3 any other point on the cubic: the brackets $\{\ \}$ are of course here necessary in order to distinguish $\{012\}$ from the determinant 012. It will be remembered that $\{012\}$ is an alternate function

$$\{012\},\ =-\{102\},\ =\{120\}, \text{ etc.}$$

If instead of 1, 2 we take the parametric points to be α, β, coordinates $(x_\alpha, y_\alpha, z_\alpha)$ and $(x_\beta, y_\beta, z_\beta)$ respectively, then the formula is

$$Q_{\alpha\beta}=Q(0;\,\alpha,\,\beta;\,3)=\{0\alpha\beta\}-\{3\alpha\beta\}.$$

Hence taking on the cubic a new point γ, coordinates $(x_\gamma, y_\gamma, z_\gamma)$ and forming the functions $Q_{\beta\gamma}$ and $Q_{\gamma\alpha}$ we have

$$\begin{aligned} Q_{\beta\gamma}+Q_{\gamma\alpha}+Q_{\alpha\beta}=&\{0\beta\gamma\}+\{0\gamma\alpha\}+\{0\alpha\beta\}\\ &-\{3\beta\gamma\}-\{3\gamma\alpha\}-\{3\alpha\beta\}.\end{aligned}$$

But by the formula No. 58, $\{0\beta\gamma\}+\{0\gamma\alpha\}+\{0\alpha\beta\}=\{\alpha\beta\gamma\}$; hence also $\{3\beta\gamma\}+\{3\gamma\alpha\}+\{3\alpha\beta\}=\{\alpha\beta\gamma\}$: and we have thus

$$Q_{\beta\gamma}+Q_{\gamma\alpha}+Q_{\alpha\beta}=0,$$

the syzygy for the cubic.

85. For the conversion, the definition of H is

$$\begin{aligned} H(0;\,1,\,2;\,3,\,6)=&\,\partial_1\,Q(0;\,1,\,2;\,3)\\ &+\partial_3\,Q(1;\,3,\,2;\,6).\end{aligned}$$

viz. this is

$$\begin{aligned} H_0(1)=H(0;\,1,\,2;\,3,\,6)=&\,\partial_1(\{012\}-\{123\})\\ &+\partial_3(\{132\}-\{326\}),\\ =&\,\partial_1\{012\}-(\partial_1+\partial_3)\{123\}-\partial_3\{326\}.\end{aligned}$$

which in virtue of $(\partial_1+\partial_2+\partial_3)\{123\}=0$ (see No. 63) becomes

$$H_0(1)=\partial_1\{012\}+\partial_2\{123\}-\partial_3\{326\}.$$

Interchanging the 0 and 1 we thence have

$$H_1(0)=\partial_0\{102\}+\partial_2\{023\}-\partial_3\{326\}.$$

Hence the difference $H_0(1)-H_1(0)$ is

$$=\partial_1\{012\}-\partial_0\{102\}+\partial_2\{123\}-\partial_2\{023\},$$

viz. this is $=(\partial_1+\partial_0)\{012\}+\partial_2(\{123\}-\{230\})$,

where the first term is $=-\partial_2\{012\}$ and the whole therefore is

$$=\partial_2(\{123\}-\{230\}-\{012\})$$

$$=-\partial_2\{301\},$$

in virtue of $\{123\}-\{230\}+\{301\}-\{012\}=0$, and is consequently $=0$.

We have thus $H_0(1)-H_1(0)=\sigma$,

the conversion-equation in the case of the cubic.

CHAPTER V. MISCELLANEOUS INVESTIGATIONS.

The Differential Symbol $d\omega$. Art. Nos. 86 and 87.

86. The definition is

$$\frac{ydz-zdy}{\frac{df}{dx}}=\frac{zdx-xdz}{\frac{df}{dy}}=\frac{xdy-ydx}{\frac{df}{dz}}=d\omega,$$

and it hence follows that we have

$$d\omega=\frac{\begin{vmatrix} dx, & dy, & dz\\ x, & y, & z\\ \lambda, & \mu, & \nu\end{vmatrix}}{\lambda\frac{df}{dx}+\mu\frac{df}{dy}+\nu\frac{df}{dz}},$$

where (λ, μ, ν) are arbitrary constants, or if we please arbitrary functions of (x, y, z): viz. the expression just written down is altogether independent of the values of λ, μ, ν: and is consequently equal to the value obtained by writing any two of these symbols $=0$, that is, the expression is equal to any one of the foregoing three equal values of $d\omega$. The expression was first given by Aronhold (1863), in the memoir presently referred to.

It is to be remarked that considering (λ, μ, ν) as the coordinates of a point, the denominator $\lambda\frac{df}{dx}+\mu\frac{df}{dy}+\nu\frac{df}{dz}$ equated to 0 is the polar $(n-1)$thic of the point λ, μ, ν in regard to the fixed curve.

If instead of λ, μ, ν we write $bc'-b'c$, $ca'-c'a$, $ab'-a'b$, where (a, b, c) (a', b', c') are constants, then the numerator is $=(ax+by+cz)(a'dx+b'dy+c'dz)-(a'x+b'y+c'z)(adx+bdy+cdz)$, or introducing ρ, σ to denote the arbitrary linear functions $ax+by+cz$ and $a'x+b'y+c'z$ respectively, the numerator is $=\rho d\sigma-\sigma d\rho$: moreover, observing that a, b, c and a', b', c' are the differential coefficients of ρ, σ in regard to the coordinates (x, y, z), the denominator is $=J(f, \rho, \sigma)$; and the value of $d\omega$ is

$$d\omega=\frac{\rho d\sigma-\sigma d\rho}{J(f, \rho, \sigma)},$$

where in accordance with a previous remark, the denominator equated to 0 is the polar $(n-1)$thic of the intersection of the lines $\rho=0$, $\sigma=0$ in regard to the fixed curve.

Obviously by taking for ρ, σ any two of the three coordinates x, y, z, we reproduce the original three forms of $d\omega$.

87. The last mentioned form of $d\omega$ suggests the expression for this symbol in the case where the fixed curve, instead of being a plane curve, is a curve of double curvature defined by two equations $f=0$, $g=0$ between the four coordinates (x, y, z, w): viz. ρ, σ being now arbitrary linear functions

$$ax+by+cz+dw, \text{ and } a'x+b'y+c'z+d'w$$

of the four coordinates, the expression is

$$d\omega=\frac{\rho d\sigma-\sigma d\rho}{J(f, g, \rho, \sigma)}:$$

and by taking for ρ, σ any two of the four coordinates x, y, z, w, we have for $d\omega$ six values which must of course be equal to each other; it is easy to verify *à posteriori* that this is so.

In the case where the curve of double curvature is not the complete intersection of two surfaces, the denominator (regarded as the Jacobian of the curve and of the arbitrary planes ρ, σ) will have a definite meaning, but what this is I do not at present consider.

The last mentioned expression for $d\omega$ will be applied further on to the case of the quadri-quadric curve $y^2+x^2=1$, $z^2+k^2x^2=1$.

Integral Formulæ. Art. Nos. 88 to 90.

88. In what precedes $d\omega$ has been used as a single symbol to denote any one of the equal differential expressions

$$\frac{ydz-zdy}{\frac{df}{dx}},=\frac{zdx-xdy}{\frac{df}{dy}},=\frac{xdy-ydx}{\frac{df}{dz}};$$

there is no quantity ω. These expressions are of the order $-(n-3)$ in the coordinates (x, y, z), and since (x, y, z) are as to their absolute magnitudes altogether arbitrary (only their ratios being determinate), a symbol such as

$$\omega, =\int d\omega, =\int \frac{ydz - zdy}{\frac{df}{dy}},$$

would, except in the case $n=3$, be altogether meaningless. In fact the integral would be

$$\int \frac{z^2 d\left(\frac{y}{z}\right)}{z^{n-1}\phi\left(\frac{x}{z}, \frac{y}{z}\right)}, =\int \frac{d\left(\frac{y}{z}\right)}{z^{n-3}\phi\left(\frac{x}{z}, \frac{y}{z}\right)},$$

where $\frac{x}{z}$ is by the equation of the fixed curve given as a function of $\frac{y}{z}$, but the other factor z^{n-3} is an absolutely indeterminate variable value, and the expression is meaningless.

But we have integrals $\int Qd\omega$, where Q is a homogeneous function of the order $n-3$ in the coordinates (x, y, z); and in particular we have such integrals where (corresponding to the forms which present themselves in the differential pure and affected theorems respectively) Q is either a rational and integral function $(x, y, z)^{n-3}$, or a rational and integral function $(x, y, z)^{n-2}$ divided by a linear function $(x, y, z)^1$: for in every such case the form of integral is

$$\int \frac{d\left(\frac{y}{z}\right)}{\phi\left(\frac{x}{z}, \frac{y}{z}\right)},$$

where $\frac{x}{z}$ is a given function of $\frac{y}{z}$, and the factor of $d\left(\frac{y}{z}\right)$ is thus a mere function of $\frac{y}{z}$. More definitely, in the integrals $\int Qd\omega$ which are considered, Q is either a minor function $(x, y, z)^{n-3}$, or it is the quotient of a major function $(x, y, z)_{12}^{n-2}$ by the linear function 012.

In the case $n=2$ there is no rational and integral function $(x, y, z)^{n-3}$, but the function may be of the form belonging to the affected theorem, unity divided by a linear function $(x, y, z)^1$; or say the integral is $\int \frac{d\omega}{ax+\beta y+\gamma z}$, where the (x, y, z) are connected by a quadric equation $(a, \ldots \between x, y, z)^2=0$: it will be shown presently that this integral is obtainable as a logarithmic function.

In the case $n=3$, we have the rational and integral function $(x, y, z)^{n-3}$, $=$ a constant, or say $=1$, so that there is here an integral $\int d\omega$: we do not call this ω, but introducing a new letter, say u, and fixing at pleasure the inferior limit of the integral, we write $u=\int d\omega$.

89. In the foregoing form $\int Qd\omega$, so long as we retain the symbol $d\omega$, there is nothing to show what is the variable in regard to which the integration is to be performed; we may for instance writing

$$d\omega=\frac{y^2 d\left(\frac{z}{y}\right)}{\frac{df}{dx}}$$

make it to be $\frac{y}{z}$, or in like manner to be any other of the six quotients. We thus cannot attribute a *value* to the inferior or superior limit of such an integral, but we may take the limits to be each of them a point on the fixed curve: for instance if 1, 0 be points on the fixed curve then the integral $\int_0^1 Qd\omega$, means the integral taken from the value at the point 0 to the value at the point 1, of the variable in regard to which the integration is performed; or when there is no expressed superior limit, then the integral is to be taken from the value for the expressed or known inferior limit to the value at the current point (x, y, z) of the variable in regard to which the integration is performed. The actual value of the integral will of course depend upon the *path* of the variable; but this is a question which is not here entered upon.

If using Cartesian Coordinates x, y, we write for instance

$$d\omega=\frac{dx}{\frac{df}{dy}}\,,\ \text{then}\ \int\frac{Qdx}{\frac{df}{dy}}$$

will denote an integral $\int \phi x\, dx$ in regard to the variable x, and the inferior and superior limits will be as usual values of x, or if there is no expressed superior limit then the integral $\int_0 \phi x\, dx$ will be the integral taken from the inferior limit x_0 to the current value x.

We may, if we please, consider the coordinates (x, y, z) as depending upon a parameter ϑ, viz. the ratios $x:y:z$ may be regarded as given functions of ϑ, and the integral $\int Qd\omega$, is then an integral $\int \Omega d\vartheta$, which taken from a constant inferior limit up to the value ϑ, which belongs to a given point 1 of the curve, is a given function of ϑ_1, or say of the point 1. But except in the case of the cubic (or generally if $p=1$), we do not have the coordinates actually given as known functions of a parameter ϑ (say they are potentially known functions of ϑ), and it is further to be noticed the functions which present themselves are functions not of a single point, but of p or more points: thus in the case of the quartic, $n=4$, $p=3$; we have $\int^1 x\,d\omega$, $\int^1 y\,d\omega$, $\int^1 z\,d\omega$, each standing for a given function of the parameter ϑ_1, but these integrals do not present themselves singly, but in combinations such as $\left(\int^1+\int^2+\int^3+\int^\xi\right)(x\,d\omega,\ y\,d\omega,\ z\,d\omega)$, say these sums of integrals are u, v, w: each of the functions u, v, w is a potentially known function of the parameters ϑ_1, ϑ_2, ϑ_3, ϑ_ξ which belong to the points 1, 2, 3, ξ respectively, and is consequently regarded as a given function of these four points.

90. Consider as before, in the case of a cubic curve, the integral $u=\int d\omega$: it will presently be seen that for the general curve as given by a cubic equation $f=0$ of any form whatever, we arrive at a form of elliptic function: but the ordinary elliptic functions sn, cn, dn connect themselves most readily with the cubic curve $y^2=x.1-x.1-k^2x$. We have here

$$d\omega=\frac{\frac{1}{2}dx}{y},=\frac{\frac{1}{2}dx}{\sqrt{x.1-x.1-k^2x}},$$

or, in the equation $u=\int d\omega$, taking the inferior limit to be 0, say

$$u=\int_0\frac{\frac{1}{2}dx}{\sqrt{x.1-x.1-k^2x}},$$

an equation which determines u as a function of x, or conversely, x as a function of u. We might thence by means of Abel's theorem as applied to the curve in question investigate the properties of the function $x=\lambda(u)$ thus arising, and so establish the theory of elliptic functions: but it is more convenient, treating the elliptic functions as known functions, to write for λu its value; viz. to take for x as given by this equation, the value $x=\operatorname{sn}^2u$: we thence have $y=\operatorname{sn}u\operatorname{cn}u\operatorname{dn}u$; viz. these values $x=\operatorname{sn}^2u$, $y=\operatorname{sn}u\operatorname{cn}u\operatorname{dn}u$, satisfy the equation $y^2=x.1-x.1-k^2x$ of the curve, and give, moreover, $d\omega=du=\frac{\frac{1}{2}dx}{y}$: and we can with these values, and the formulæ for elliptic functions, verify any results given by Abel's theorem. This will be done in considerable detail: but at present I wish only to remark that the formulæ give the coordinates x, y of a point on the cubic curve expressed as one-valued functions of a parameter or argument u: but that this argument u is not a one-valued function of the coordinate x, or even of the coordinates x, y of the given point on the curve: say the argument u has not a unique value for a given point (x, y) of the curve. There are in fact an infinity of values $u = u_0 + 2mK + 2m'iK'$, where m, m' are any positive or negative integers: that this is so, depends on the multiplicity of values, according to the different paths of the variable, of the integral $u = \int_0 \frac{\frac{1}{2}dx}{\sqrt{x.\overline{1-x}.\overline{1-k^2x}}}$; or, regarding the elliptic functions as known functions, it depends upon the double periodicity of these functions.

Aronhold's Quadric Integral. Art. Nos. 91 to 93.

91. I reproduce the investigation contained in Aronhold's paper "Ueber eine neue algebraische Behandlungsweise u. s. w., Crelle, t. lxi, 1863, pp. 95–145. We take f the general quadric function $(a, b, c, f, g, h \between x, y, z)^2$; $\alpha x + \beta y + \gamma z$ an arbitrary linear function of x, y, z: the theorem is $\frac{d\omega}{\alpha x + \beta y + \gamma z}$ = differential of logarithm of an algebraic function of (x, y, z); viz. taking (ξ, η, ζ) for the coordinates of either of the points of intersection of the line $\alpha x + \beta y + \gamma z = 0$ with the quadric $(a, \ldots \between x, y, z)^2 = 0$, and writing also

$$\Omega^2 = -(bc - f^2,\ ca - g^2,\ ab - h^2,\ gh - af,\ hf - bg,\ fg - ch \between \alpha, \beta, \gamma)^2,$$

then the theorem is

$$\frac{d\omega}{\alpha x + \beta y + \gamma z} = \frac{1}{\Omega}\, d.\log \frac{(a, \ldots \between x, y, z \between \xi, \eta, \zeta)}{\alpha x + \beta y + \gamma z},$$

or, what is the same thing,

$$\int \frac{d\omega}{\alpha x + \beta y + \gamma z} = \frac{1}{\Omega} \log \frac{(a, \ldots \between x, y, z \between \xi, \eta, \zeta)}{\alpha x + \beta y + \gamma z} + \text{const.}$$

It is to be observed in regard to this equation that the two sides respectively are in regard to (α, β, γ) homogeneous functions of the degree -1, and in regard to (ξ, η, ζ) homogeneous of the degree 0; viz. on the right-hand side the effect of a change in the absolute magnitudes of ξ, η, ζ, say the change into $k\xi$, $k\eta$, $k\zeta$, is merely to change by $\log k$ the constant of integration.

It is to be remarked also that the equation $(a, \ldots \between \xi, \eta, \zeta \between x, y, z) = 0$, represents the tangent to the conic at the point (ξ, η, ζ) of intersection with the line $\alpha x + \beta y + \gamma z = 0$; calling the linear function in question T, the value of the integral is $\frac{1}{\Omega}\log\frac{T}{\alpha x+\beta y+\gamma z}$; if (ξ_1, η_1, ζ_1), (ξ_2, η_2, ζ_2) are the coordinates of the two points of intersection respectively, then in passing from one of these to the other we change the sign of the radical Ω, and the two values thus are $\frac{1}{\Omega}\log\frac{T_1}{\alpha x+\beta y+\gamma z}$ and $-\frac{1}{\Omega}\log\frac{T_2}{\alpha x+\beta y+\gamma z}$, these must differ by a constant only; viz. we should have $\log\frac{T_1T_2}{(\alpha x+\beta y+\gamma z)^2} =$ a const. And in fact T_1, T_2 being the tangents to the conic f at its intersections with the line $\alpha x+\beta y+\gamma z=0$, we have it is clear $f=\lambda T_1T_2+\mu(\alpha x+\beta y+\gamma z)^2$, that is, (x, y, z) referring to a point of the conic $f=0$, we have $\frac{T_1T_2}{(\alpha x+\beta y+\gamma z)^2} =$ a constant, which is right.

92. We require the coordinates (ξ, η, ζ) of an intersection: these are determined by the equations $\alpha\xi+\beta\eta+\gamma\zeta=0$, $(a, \ldots \between \xi, \eta, \zeta)^2=0$, or as these may be written

$$\begin{array}{l}\alpha\xi \qquad\qquad\qquad + \beta\eta \qquad\qquad\qquad + \gamma\zeta = 0,\\ (a\xi+h\eta+g\zeta)\,\xi+(h\xi+b\eta+f\zeta)\,\eta+(g\xi+f\eta+c\zeta)\,\zeta=0,\end{array}$$

we have thence ξ, η, ζ proportional to the determinants

$$\left|\begin{array}{ccc} a\xi+h\eta+g\zeta, & h\xi+b\eta+f\zeta, & g\xi+f\eta+c\zeta \\ \alpha \quad , & \beta \quad , & \gamma \end{array}\right|,$$

say these determinants are $\Omega\xi, \Omega\eta, \Omega\zeta$, where Ω is a value as yet undetermined. The equations are $\gamma(h\xi+b\eta+f\zeta)-\beta(g\xi+f\eta+c\zeta)-\Omega\xi=0$, etc., viz. these are

$$\begin{array}{l}(\gamma h-\beta g-\Omega)\,\xi+(\gamma b-\beta f)\,\eta+(\gamma f-\beta c)\,\zeta=0\\ (\alpha g-\gamma a)\,\xi+(\alpha f-\gamma h-\Omega)\,\eta+(\alpha c-\gamma g)\,\zeta=0\\ (\beta a-\alpha h)\,\xi+(\beta h-\alpha b)\,\eta+(\beta g-\alpha f-\Omega)\,\xi=0,\end{array}$$

eliminating (ξ, η, ζ) we have an equation which may be written

$$\left|\begin{array}{lll} A-\Omega, & B \quad , & C \\ A' \quad , & B'-\Omega, & C' \\ A'' \quad , & B'' \quad , & C''-\Omega \end{array}\right| = 0.$$

that is

$$\left|\begin{array}{lll} A\,, & B\,, & C \\ A', & B', & C' \\ A'', & B'', & C'' \end{array}\right| -\Omega(B'C''-B''C'+C''A-CA''+AB'-A'B) +\Omega^2(A+B'+C'')-\Omega^3=0.$$

We find very easily that the determinant, and $A+B'+C''$ are each $=0$; and the equation thus reduces itself to

$$\Omega^2=B'C''-B''C'+C''A-CA''+AB'-A'B,$$

or substituting for A, B, etc. their values

$$\Omega^2=-(bc-f^2, \ldots \between \alpha, \beta, \gamma)^2,$$

and this being so, the ratios of ξ, η, ζ are determined by means of any two of the foregoing linear equations.

93. We may now verify the theorem; in the general expression for $d\omega$ writing for λ, μ, ν the values ξ, η, ζ, the equation to be verified becomes

$$\frac{\begin{vmatrix} dx, & dy, & dz \\ x\ , & y\ , & z \\ \xi\ , & \eta\ , & \zeta \end{vmatrix}}{(\alpha x+\beta y+\gamma z)(a, \ldots \between x, y, z \between \xi, \eta, \zeta)} = \frac{1}{\Omega}\left\{\frac{(a, \ldots \between \xi, \eta, \zeta \between dx, dy, dz)}{(a, \ldots \between \xi, \eta, \zeta \between x, y, z)} - \frac{\alpha dx+\beta dy+\gamma dz}{\alpha x+\beta y+\gamma z}\right\}.$$

viz. this is

$$\Omega \begin{vmatrix} dx, & dy, & dz \\ x\ , & y, & z \\ \xi\ , & \eta, & \zeta \end{vmatrix} \begin{aligned} &= (a, \ldots \between \xi, \eta, \zeta \between dx, dy, dz).(\alpha x+\beta y+\gamma z) \\ &\quad - (a, \ldots \between \xi, \eta, \zeta \between x\ , y\ , z).(\alpha dx+\beta dy+\gamma dz). \end{aligned}$$

Here on the right-hand side the coefficient of dx is

$$\begin{aligned} &(a\xi+h\eta+g\zeta)(\alpha x+\beta y+\gamma z) \\ &-\alpha\{(a\xi+h\eta+g\zeta)x+(h\xi+b\eta+f\zeta)y+(g\xi+f\eta+c\zeta)z\} \end{aligned}$$

which is

$$\begin{aligned} &= y\{\beta(a\xi+h\eta+g\zeta)-\alpha(h\xi+b\eta+f\zeta)\} \\ &\quad - z\{\alpha(g\xi+f\eta+c\zeta)-\gamma(a\xi+h\eta+g\zeta)\}, \\ &= y.\Omega\zeta - z.\Omega\eta, \\ &= \Omega(y\zeta - z\eta), \end{aligned}$$

which is right; and similarly the coefficients of dy and dz have the same values on the two sides of the equation respectively.

Aronhold's Quadric Integral deduced from the Affected Theorem.

Art. Nos. 94 to 98.

94. Let the fixed curve be a conic, say $f=\frac{1}{2}(a, b, c, f, g, h \between x, y, z)^2, =0$: and let the variable curve be a line meeting the conic in the points 3 and 4. The affected theorem is

$$\Sigma\frac{12 d\omega}{012} = -\frac{\delta 134}{134}\frac{\delta 234}{234},$$

where (x_1, y_1, z_1) and (x_2, y_2, z_2) being the coordinates of the points 1 and 2 respectively, 12 denotes the constant $(a, \ldots \between x_1, y_1, z_1 \between x_2, y_2, z_2)$: and 012, etc. denote determinants as usual.

The left-hand side is here

$$12\left\{\frac{d\omega_3}{312}+\frac{d\omega_4}{412}\right\}:$$

on the right-hand side δ refers to the variation of the constants of ϕ, that is to the variations of the points 3 and 4; or we may write $\delta = d_3 + d_4$; the points 3, 4 are independent, and the equation, being satisfied at all, must be satisfied separately in regard to the variations of 3, and in regard to the variations of 4: we must therefore have

$$12\frac{d\omega_3}{312} = -\frac{d_3 134}{134} + \frac{d_3 234}{234};$$

and the like equation obtained herefrom by the interchange of the numbers 3 and 4.

95. The equation just written down relates to any four points 1, 2, 3, 4 of the conic; and if for 3, 4 we write 0, 3 respectively, it becomes

$$12\frac{d\omega}{012} = -\frac{d.031}{031} + \frac{d.023}{023},$$

which relates to the points 0, 1, 2, 3 of the conic: writing as before 023, 031, $012 = \rho, \sigma, \tau$, this equation is

$$12\frac{d\omega}{\tau} = -\frac{d\sigma}{\sigma} + \frac{d\rho}{\rho},$$

which may be verified as follows: the equation of the conic is $f = 23.\sigma\tau + 31.\tau\rho + 12.\rho\sigma, = 0$: we have $d\omega = \dfrac{\rho d\sigma - \sigma d\rho}{\frac{df}{d\tau}}$, where $\dfrac{df}{d\tau} = 23.\sigma + 31\rho, = -\dfrac{12.\rho\sigma}{\tau}$, that is, $d\omega = \dfrac{12}{\tau}\left(-\dfrac{d\sigma}{\sigma} + \dfrac{d\rho}{\rho}\right)$, the equation in question.

96. We have as a property of any four points 0, 1, 2, 3 of a conic $\dfrac{23}{123.023} = \dfrac{-01}{012.031}$, or say $\dfrac{23}{\Delta.\rho} = \dfrac{-01}{\sigma\tau}$, that is $\dfrac{23}{\Delta}\dfrac{\sigma}{\rho} = -\dfrac{01}{\tau}$; hence considering 0 as a variable point, and differentiating the logarithms,

$$-d\log\frac{01}{\tau} = -\frac{d\sigma}{\sigma} + \frac{d\rho}{\rho},$$

and the foregoing equation $12\dfrac{d\omega}{\tau} = -\dfrac{d\sigma}{\sigma} + \dfrac{d\rho}{\rho}$, thus becomes $12\dfrac{d\omega}{\tau} = -d\log\dfrac{01}{\tau}$, or restoring for τ its value 012,

$$12\frac{d\omega}{012} = -d\log\frac{01}{012}.$$

Taking now $\alpha x + \beta y + \gamma z = 0$ for the equation of the line 012; this meets the conic in the points 1, 2, coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively: and we have

$$\alpha, \beta, \gamma = y_1 z_2 - y_2 z_1,\ z_1 x_2 - z_2 x_1,\ x_2 y_1 - x_1 y_2,$$

$$12 = (a, \ldots \between x_1, y_1, z_1 \between x_2, y_2, z_2),$$

and from this last value

$$\overline{12}^2 = \{(a, \ldots)(x_1, y_1, z_1)(x_2, y_2, z_2)\}^2 - (a, \ldots)(x_1, y_1, z_1)^2 . (a, \ldots)(x_2, y_2, z_2)^2$$

(the second term being of course $= 0$), viz. this is

$$\begin{aligned}\overline{12}^2 &= -(bc - f^2, \ldots)(y_1 z_2 - y_2 z_1,\ z_1 x_2 - z_2 x_1,\ x_1 y_2 - x_2 y_1)^2 \\ &= -(bc - f^2, \ldots)(\alpha, \beta, \gamma)^2,\end{aligned}$$

or say $12 = -\Omega$, if $\Omega^2 = -(bc - f^2, \ldots)(\alpha, \beta, \gamma)^2$ as before: and the equation thus is

$$\frac{\Omega d\omega}{\alpha x + \beta y + \gamma z} = d \log \frac{(a \ldots)(x_1, y_1, z_1)(x, y, z)}{\alpha x + \beta y + \gamma z},$$

or finally, writing (ξ, η, ζ) instead of (x_1, y_1, z_1) to denote the coordinates of one or other of the intersections of the line $\alpha x + \beta y + \gamma z = 0$ with the conic, the equation becomes

$$\frac{\Omega d\omega}{\alpha x + \beta y + \gamma z} = d \log \frac{(a, \ldots)(\xi, \eta, \zeta)(x, y, z)}{\alpha x + \beta y + \gamma z},$$

which is Aronhold's quadric integral.

97. (The foregoing property, which may also be written

$$\frac{23}{\overline{023} . \overline{123}} = \frac{01}{\overline{201} . \overline{301}},$$

is verified very simply in the case of four points 0, 1, 2, 3 of a circle: in fact

$$23 = x_2 x_3 + y_2 y_3 - 1, = \cos 23 - 1, = -2 \sin^2 \tfrac{1}{2} 23,$$
$$023 = 2 \sin \tfrac{1}{2} 23 \sin \tfrac{1}{2} 30 \sin \tfrac{1}{2} 02;$$

and so for the other like expressions; each side of the equation is thus reduced to $1 \div \sin \frac{1}{2} 02 \sin \frac{1}{2} 03 \sin \frac{1}{2} 12 \sin \frac{1}{2} 13$).

98. In particular if the conic is taken to be the circle $x^2 + y^2 - 1 = 0$, then for the coordinates $\left(\frac{\xi}{\zeta}, \frac{\eta}{\zeta}\right)$ of the intersections with the line $\alpha x + \beta y + \gamma = 0$, we have

$$\begin{aligned}&\Omega\xi + \gamma\eta + \beta\zeta = 0, \quad \text{giving } \Omega^2 = \alpha^2 + \beta^2 + \gamma^2 \\ &\gamma\xi - \Omega\eta + \alpha\zeta = 0, \\ &\beta\xi - \alpha\eta + \Omega\zeta = 0,\end{aligned}$$

and then

$$\begin{aligned}\xi : \eta : \zeta &= -\beta^2 + \gamma^2 : \alpha\beta - \gamma\Omega : \alpha\gamma + \beta\Omega \\ &= -\alpha\beta - \gamma\Omega : \alpha^2 - \gamma^2 : \beta\gamma + \alpha\Omega \\ &= \alpha\gamma + \beta\Omega : \beta\gamma - \alpha\Omega : -\alpha^2 - \beta^2,\end{aligned}$$

and the formula then becomes

$$\int \frac{dx}{(\alpha x + \beta\sqrt{1 - x^2} + \gamma)\sqrt{1 - x^2}} = \frac{1}{\sqrt{\alpha^2 + \beta^2 - \gamma^2}} \log \frac{\xi x + \eta\sqrt{1 - x^2} - \zeta}{\alpha x + \beta\sqrt{1 - x^2} + \gamma},$$

or, retaining Ω, y for the values $\sqrt{\alpha^2 + \beta^2 - \gamma^2}$, and $\sqrt{1 - x^2}$, as this may also be written

$$= \frac{1}{\Omega} \log \frac{\gamma(\alpha x + \beta y + \gamma) + \Omega(\beta x - \alpha y + \Omega)}{\alpha x + \beta y + \gamma}$$

The form of the integral is still such that the value is not very readily obtainable by ordinary methods: the value just written down can of course be verified, but the verification is scarcely easier than for the original more general form.

In the very particular case $\alpha = 0$, $\beta = 0$, $\gamma = 1$, we have $\Omega = i$; $\xi : \eta : \zeta = 1 : -i : 0$ and the formula becomes

$$\int \frac{dx}{\sqrt{1-x^2}} = \frac{1}{i} \log\,(x - iy) : \text{ viz. this is}$$

$$\sin^{-1} x = \tfrac{1}{2}\pi + \frac{1}{i} \log\,(x - i\sqrt{1-x^2}),$$

which is right; for putting $\sin^{-1} x = u$, and therefore $x = \sin u$, the equation becomes $i(u - \tfrac{1}{2}\pi) = \log.\,(\sin u - i\cos u)$: that is $\cos(u - \tfrac{1}{2}\pi) + i\sin(u - \tfrac{1}{2}\pi) = \sin u - i\cos u$.

Fixed Curve a Cubic: *the Parametric Points* 1, 2 *consecutive points on the Curve.* Art. Nos. 99 to 106.

99. The major function $(x, y, z)^1_{12}$ is taken to be $= \dfrac{1^2 2.023 - 12^2.013}{123}$, so that calling the differential $Q d\omega$ we have

$$Q = \frac{1^2 2.023 - 12^2.013}{123.012},$$

and it is required to find what this becomes when 1, 2 are consecutive points on the curve, or what is the same thing when the line 012 is a tangent at the point 1.

I take for convenience the cubic to be $f, = \tfrac{1}{3}(x^3 + y^3 + z^3), = 0$. The coordinates of 1 are (x_1, y_1, z_1), those of 2 are $(x_1 + \delta x_1, y_1 + \delta y_1, z_1 + \delta z_1)$, or as for shortness I write them $(x_1 + \alpha, y_1 + \beta, z_1 + \gamma)$, where α, β, γ are considered as infinitesimals of the first order: this being so, the denominator of Q is at once seen to be of the second order; it will appear that the numerator is of the third order; whence, Q is of the first order.

100. We have $d\omega = \dfrac{ydz - zdy}{x^2}, = \dfrac{zdx - xdz}{y^2} = \dfrac{xdy - ydx}{z^2}$, and in analogy herewith we may write $\delta\omega_1 = \dfrac{y_1\gamma - z_1\beta}{x_1^2}, = \dfrac{z_1\alpha - x_1\gamma}{y_1^2}, = \dfrac{x_1\beta - y_1\alpha}{z_1^2}$; this being so we have

$$012 = \begin{vmatrix} x, & y, & z \\ x_1 & y_1 & z_1 \\ \alpha & \beta & \gamma \end{vmatrix} = (xx_1^2 + yy_1^2 + zz_1^2)\,\delta\omega_1 = 01^2.\delta\omega_1,$$

and similarly $312 = 31^2.\delta\omega_1$.

Moreover

$$023 = \begin{vmatrix} x, & y, & z \\ x_1 & y_1 & z_1 \\ x_3 & y_3 & z_3 \end{vmatrix} + \begin{vmatrix} x\,, & y\,, & z \\ \alpha\,, & \beta\,, & \gamma \\ x_3, & y_3, & z_3 \end{vmatrix} = 013 + 0\delta 13,$$

as the second term may be written; moreover

$$1^2 2 = x_1^2(x_1+\alpha) + y_1^2(y_1+\beta) + z_1^2(z_1+\gamma), \; = \; \alpha x_1^2+\beta y_1^2+\gamma z_1^2,$$

$$12^2 = x_1(x_1+\alpha)^2 + y_1(y_1+\beta)^2 + z_1(z_1+\gamma)^2, \; = 2(\alpha x_1^2+\beta y_1^2+\gamma z_1^2)+\alpha^2 x_1+\beta^2 y_1+\gamma^2 z_1,$$

and hence

$$\begin{aligned} 1^2 2.023 - 12^2.013 = & \quad (\alpha x_1^2 + \beta y_1^2 + \gamma z_1^2) \quad (013 + 0\delta 13) \\ & - [2(\alpha x_1^2 + \beta y_1^2 + \gamma z_1^2) + (\alpha^2 x_1 + \beta^2 y_1 + \gamma^2 z_1)]\, 013 \\ = & - \quad [(\alpha x_1^2 + \beta y_1^2 + \gamma z_1^2) + (\alpha^2 x_1 + \beta^2 y_1 + \gamma^2 z_1)]\, 013 \\ & + \quad (\alpha x_1^2 + \beta y_1^2 + \gamma z_1^2).0\delta 13, \end{aligned}$$

or reducing by $3(\alpha x_1^2 + \beta y_1^2 + \gamma z_1^2) + 3(\alpha^2 x + \beta^2 y + \gamma^2 z) + (\alpha^3 + \beta^3 + \gamma^3) = 0$,
this is $\qquad = + \tfrac{1}{3}(\alpha^3 + \beta^3 + \gamma^3)\, 013 - (\alpha^2 x_1 + \beta^2 y_1 + \gamma^2 z_1)\, 0\delta 13,$
which is of the third order.

101. We may show that each of the terms contains the factor $(\delta\omega_1)^2$: we have in fact

$$y_1 z_1(\delta\omega_1)^2 = \alpha\beta\frac{x_1}{y_1} - \alpha^2 - \beta\gamma\frac{x_1^2}{y_1 z_1} + \gamma\alpha\frac{x_1}{z_1},$$

$$z_1 x_1(\delta\omega_1)^2 = \beta\gamma\frac{y_1}{z_1} - \beta^2 - \gamma\alpha\frac{y_1^2}{z_1 x_1} + \alpha\beta\frac{y_1}{x_1},$$

$$x_1 y_1(\delta\omega_1)^2 = \gamma\alpha\frac{z_1}{x_1} - \gamma^2 - \alpha\beta\frac{z_1^2}{x_1 y_1} + \beta\gamma\frac{z_1}{y_1},$$

and hence first multiplying by α, β, γ and adding we have

$$\begin{aligned} (\alpha y_1 z_1 + \beta z_1 x_1 + \gamma x_1 y_1)(\delta\omega_1)^2 = & \frac{\beta\gamma}{y_1 z_1}(\beta y_1^2 + \gamma z_1^2) + \frac{\gamma\alpha}{z_1 x_1}(\gamma z_1^2 + \alpha x_1^2) + \frac{\alpha\beta}{x_1 y_1}(\alpha x_1^2 + \beta y_1^2) \\ & - \alpha\beta\gamma\left(\frac{x_1^2}{y_1 z_1} + \frac{y_1^2}{z_1 x_1} + \frac{z_1^2}{x_1 y_1}\right) \\ & - (\alpha^3 + \beta^3 + \gamma^3). \end{aligned}$$

But in virtue of $\alpha x_1^2 + \beta y_1^2 + \gamma y_1^2 = 0$, the first line becomes = the second line, or the two together are $= -2\alpha\beta\gamma\left(\frac{x_1^2}{y_1 z_1} + \frac{y_1^2}{z_1 x_1} + \frac{z_1^2}{x_1 y_1}\right)$ which is = 0 in virtue of $x_1^3 + y_1^3 + z_1^3 = 0$; hence the equation is

$$(\alpha y_1 z_1 + \beta z_1 x_1 + \gamma x_1 y_1)(\delta\omega_1)^2 = -(\alpha^3 + \beta^3 + \gamma^3),$$

the required expression for the first term.

102. Again, multiplying by x_1, y_1, z_1, and adding we have

$$\begin{aligned} 3x_1 y_1 z_1(\delta\omega_1)^2 = & \frac{\beta\gamma}{y_1 z_1}(y_1^3 + z_1^3 - x_1^3) + \frac{\gamma\alpha}{z_1 x_1}(z_1^3 + x_1^3 - y_1^3) + \frac{\alpha\beta}{x_1 y_1}(x_1^3 + y_1^3 - z_1^3) \\ & - (\alpha^2 x_1 + \beta^2 y_1 + \gamma^2 z_1), \end{aligned}$$

where in virtue of $x_1^3 + y_1^3 + z_1^3 = 0$, the first line is

$$= -\frac{2}{x_1 y_1 z_1}(\beta\gamma x_1^4 + \gamma\alpha y_1^4 + \alpha\beta z_1^4),$$

and this again is $=-2(\alpha^2x_1+\beta^2y_1+\gamma^2z_1)$: (in fact we have identically

$$\begin{aligned}x_1y_1z_1(\alpha^2x_1+\beta^2y_1+\gamma^2z_1)=&(\alpha x_1^2+\beta y_1^2+\gamma z_1^2)(\alpha y_1z_1+\beta z_1x_1+\gamma x_1y_1)\\&-(\beta\gamma x_1+\gamma\alpha y_1+\alpha\beta z_1)(x_1^3+y_1^3+z_1^3)\\&+(\beta\gamma x_1^4+\gamma\alpha y_1^4+\alpha\beta z_1^4)\end{aligned}$$

which in virtue of $\alpha x_1^2+\beta y_1^2+\gamma z_1^2=0$, and $x_1^3+y_1^3+z_1^3=0$ becomes

$$x_1y_1z_1(\alpha^2x_1+\beta^2y_1+\gamma^2z_1)=(\beta\gamma x_1^4+\gamma\alpha y_1^4+\alpha\beta z_1)).$$

Hence the equation is $3x_1y_1z_1(\delta\omega_1)^2=-3(\alpha^2x_1+\beta^2y_1+\gamma^2z_1)$, or finally

$$x_1y_1z_1.(\delta\omega_1)^2=-\quad(\alpha^2x_1+\beta^2y_1+\gamma^2z_1),$$

the required expression for the second term.

103. Writing for shortness $\alpha y_1z_1+\beta z_1x_1+\gamma x_1y_1=\delta(x_1y_1z_1)$ we have

$$1^22.023-12^2.013=\{-\tfrac{1}{3}\delta(x_1y_1z_1)\,013+x_1y_1z_1.0\delta13\}(\delta\omega_1)^2,$$

and hence dividing by 012.123, $=01^2, 31^2.(\delta\omega_1)^2$,

we have $$Q=\frac{1^22.023-12^2.013}{012.123}=\frac{-\frac{1}{3}\delta(x_1,y_1,z_1).013+x_1y_1z_1.0\delta13}{01^2.31^2}.$$

But this can be further reduced: the numerator multiplied by 3, is

$$=-(\alpha y_1z_1+\beta z_1x_1+\gamma x_1y_1)\begin{vmatrix}x, & y, & z\\ x_1, & y_1, & z_1\\ x_3, & y_3, & z_3\end{vmatrix}+3x_1y_1z_1\begin{vmatrix}x, & y, & z\\ \alpha, & \beta, & \gamma\\ x_3, & y_3, & z_3\end{vmatrix},$$

which is $$=\begin{vmatrix}x & , & y & , & z\\ x_1(y_1^3-z_1^3), & & y_1(z_1^3-x_1^3), & & z_1(x_1^3-y_1^3)\\ x_3 & , & y_3 & , & z_3\end{vmatrix}d\omega_1,$$

where $x_1(y_1^3-z_1^3)$, $y_1(z_1^3-x_1^3)$, $z_1(x_1^3-y_1^3)$ are the coordinates of the tangential of the point 1 in regard to the cubic, viz. the point of intersection of the tangent at 1 with the cubic. The determinant may for shortness be called $0t13$; and we thus have $$Q=\frac{1^22.023-12^2.013}{012.123}=\frac{\frac{1}{3}\delta\omega_1}{31^2}\,\frac{0t13}{01^2},$$

where observe that $01^2=0$ is the equation of the tangent at the point 0: and $0t13=0$, is the equation of the line joining the tangential of 1 with the arbitrary point 3.

104. The identity just referred to is proved very easily: comparing on each side the coefficient of yz_3-y_3z, the factor x_1 divides out and we ought to have

$$-(\alpha y_1z_1+\beta z_1x_1+\gamma x_1y_1)+3y_1z_1\alpha=(y_1^3-z_1^3)\,\delta\omega_1,$$

that is $$(y_1^3-z_1^3)\,\delta\omega_1=2\alpha y_1z_1-\beta z_1x_1-\gamma x_1y_1,$$

and in fact from $y_1^2\delta\omega_1=z_1\alpha-x_1\gamma$, $z_1^2\delta\omega_1=x_1\beta-y_1\alpha$, we have

$$(y_1^3-z_1^3)\,\delta\omega_1=y_1(z_1\alpha-x_1\gamma)-z_1(x_1\beta-y_1\alpha),$$

which is the value in question: similarly the coefficients of zx_3-z_3x, xy_3-x_3y are equal on the two sides; and the equation is thus verified.

105. The proof has been given in regard to the particular cubic $x^3+y^3+z^3=0$; but it might have been given for the canonical form $x_3+y_3+z_3+6lxyz=0$: and from the invariantive form it is clear that the result in fact applies to any cubic whatever. The result is an important one: we see by it that when the points 1 and 2 are consecutive points on the curve we must in place of the differential $Qd\omega$, which is evanescent, consider a new form $\frac{0t13}{01^2}\,d\omega_1$, where, as already remarked, the denominator represents the tangent at the point 1, and the numerator the line joining the tangential of this point with the point 3.

106. We have $\quad \{023\}+\{031\}+\{012\}=\{123\}$,

or writing this in the form

$$\{012\}-\{312\}+\{023\}-\{013\}=0,$$

suppose 2 is here the consecutive point $1+\delta 1$, $\{012\}-\{312\}, = \dfrac{1^22.023-12^2.013}{012.312}$ becomes $=\{0t13\}\delta\omega_1$, we have also $\{023\}=\{013\}+\partial_1\{013\}\delta\omega_1$,

and the result is $\qquad -\{0t13\}+\partial_1 013=0$,

that is $\partial_1\{013\}=\{0t13\}$, | (in case of cubic $x^3+y^3+z^3+6lxyz=0$) is $=$

$$\frac{-\begin{vmatrix} x, & y, & z \\ x_1(y_1^3-z_1^3), & y_1(z_1^3-x_1^3), & z_1(x_1^3-y_1^3) \\ x_3, & y_3, & z_3 \end{vmatrix}}{3(x_1^2x+y_1^2y+z_1^2z)(x_1^2x_3+y_1^2y_3+z_1^2z_3)},$$

i. e. the differential coefficient of $\{013\}$ in regard to the parametric point 1, is $=\{0t13\}$ the symbol for the case where the parametric line is the tangent at 1.

Fixed Curve a Cubic: the Parametric Points corresponding points.
Art. Nos. 107 to 110.

107. The parametric points 1, 2 are taken to be corresponding points, that is such that the tangents at these points meet at a point, say 3, on the cubic. We may from 3 draw two other tangents, touching the cubic, say at the points 1′ and 2′. The four points 1, 2, 1′, 2′ are then such that the lines 12, 1′2′ meet in a point, say 4, of the cubic; and moreover 3, 4 are corresponding points.

We may take $(x, y, z), =(1, 0, 0), (0, 1, 0), (0, 0, 1)$ for the coordinates of the points 1, 2, 3 respectively: $x=0$, $y=0$ are thus the equations of the lines 32, 31 respectively, and $z=0$ is the equation of the line 12, viz. we have $z=012$. Taking $x-M_1y=0$, $x-M_2y=0$, for the equations of the tangents 31′, 32′ respectively, and $\zeta=0$ for the equation of the line 1′2′ joining their points of contact (ζ a properly determined linear function of (x, y, z)), it is to be shown that the differential $Qd\omega$ may be taken to be $\frac{\zeta d\omega}{z}$, and that this is $= \frac{1}{2}\left(\frac{dx}{x} - \frac{dy}{y}\right)$: the affected theorem thus assumes a special form, which will be noticed.

108. The cubic passes through the points $(x = 0, z = 0)$ and $(y = 0, z = 0)$, the tangents at these points being $x = 0$, and $y = 0$ respectively: also through the point $x = 0$, $y = 0$: its equation thus is

$$f, = gz^2x + 2lzxy + iz^2y + hx^2y + kxy^2, = 0,$$

and writing

$$d\omega = \frac{xdy - ydx}{\frac{df}{dz}}$$

we have

$$\frac{df}{dz} = 2(gzx + lxy + izy),$$

which from the equation of the curve written in the form

$$z(gzx + lxy + izy) + xy(hx + ky + lz) = 0,$$

or say

$$z(gzx + lxy + izy) + xy\zeta = 0,$$

becomes

$$= \frac{-2xy\zeta}{z},$$

and we thus have

$$d\omega = \frac{-z}{2xy\zeta}(xdy - ydx), = \frac{1}{2}\frac{z}{\zeta}\left(\frac{dx}{x} - \frac{dy}{y}\right),$$

where $\zeta = hx + ky + lz$. To find the meaning of ζ, observe that the line $x - My = 0$, meets the curve in the point $(x = 0, y = 0)$, and in two other points determined by the equation

$$z^2(gm + i) + 2zy\,.\,lM + y^2(hM^2 + kM) = 0;$$

this line will be a tangent if

$$(gM + i)(hM + k) - l^2M = 0,$$

and we then have at the point of contact $(hM + k)y + lz = 0$; and writing this in the form $hx + ky + lz = 0$, we see that the equation $\zeta = 0$ is satisfied at the point of contact of each of the two tangents $x - M_1y = 0$, $x - M_2y = 0$; viz. $\zeta = 0$ is the equation of the line joining the two points of contact. Moreover, from the equation of the curve written in the foregoing form

$$z(gzx + lxy + izy) + xy\zeta = 0,$$

it appears that the lines $z = 0$, $\zeta = 0$, meet on the curve; or what is the same thing, that the line $\zeta = 0$ passes through the residue of the parametric points 1, 2.

109. The function ζ at 1 becomes $=h$, and this is the value of $3.1^2 2$; in fact

$$\begin{aligned} 3.1^2 2 &= \left(x_2\frac{d}{dx_1}+y_2\frac{d}{dy_1}+z_2\frac{d}{dz_1}\right)f_1, \quad (x_2, y_2, z_2)=(0, 1, 0), \\ &=\frac{df_1}{dy_1}, \\ &= 2lz_1x_1+iz_1^2+hx_1^2+2kx_1y_1, \quad (x_1, y_1, z_1)=(1, 0, 0), \\ &=h. \end{aligned}$$

We have thus ζ, satisfying the required conditions for the major function: and the differential $Qd\omega$ may therefore be taken to be $=\frac{\zeta}{z}d\omega$, that is we have

$$Qd\omega=\tfrac{1}{2}\left(\frac{dx}{x}-\frac{dy}{y}\right).$$

The affected theorem thus becomes

$$\Sigma\tfrac{1}{2}\left(\frac{dx}{x}-\frac{dy}{y}\right)=-\frac{\delta\varphi_1}{\varphi_1}+\frac{\delta\varphi_2}{\varphi_2}.$$

110. The meaning of this will be better understood from the integral form. Integrating each side, and assuming that the superior limits are given by a line ϕ which cuts the cubic in the points 4, 5, 6, and the inferior limits by a line ψ which cuts the cubic in the points 7, 8, 9, we find

$$\log\frac{x_4x_5x_6}{x_7x_8x_9}-\log\frac{y_4y_5y_6}{y_7y_8y_9}=2\log\frac{\varphi_2}{\psi_2}\,\frac{\psi_1}{\varphi_1},$$

that is

$$\frac{x_4x_5x_6\ y_4y_5y_6}{x_7x_8x_9\ y_7y_8y_9}=\left(\frac{\psi_2\ \varphi_1}{\psi_2\ \varphi_1}\right)^2,$$

where ϕ_1, ψ_1, ϕ_2, ψ_2 denote the values of the linear functions ϕ, ψ at the points 1 and 2 respectively. We have a cubic cut by the lines ϕ, ψ, x, y in the points 4, 5, 6; 7, 8, 9; 2, 2′, 3 and 1, 1′, 3 respectively: where for the moment 1′, 2′ are written to denote the points on the curve consecutive to 1 and 2 respectively. Hence by a known theorem in transversals

$$\left(\frac{x}{y}\right)_{456}\div\left(\frac{x}{y}\right)_{789}=\left(\frac{\varphi}{\psi}\right)_{22'3}\div\left(\frac{\varphi}{\psi}\right)_{11'3}, \text{ that is}$$

$$\frac{x_4x_5x_6\ y_7y_8y_9}{x_7x_8x_9\ y_4y_5y_6}=\frac{\psi_1\psi_{1'}\psi_3\cdot\varphi_2\varphi_{2'}\varphi_3}{\psi_2\psi_{2'}\psi_3\cdot\varphi_1\varphi_{1'}\varphi_3},$$

which dividing out the $\phi_3\psi_3$, and writing 1, 2 in place of 1′, 2′, becomes

$$=\left(\frac{\varphi_2\psi_1}{\varphi_1\psi_2}\right)^2,$$

agreeing with the result just obtained.

Aronhold's Cubic Transformation. Art. Nos. 111 to 119.

111. This was obtained in the paper "Algebraische Reduction des Integrals $\int F(x, y)\,dx$, u. s. w.," Berl. Monatsb., April, 1861, pp. 462–468. I give in the first place the analytical results, independently of the general theory, with the values for the canonical form $f, = \frac{1}{3}(x^3 + y^3 + z^3 + 6lxyz), = 0$, of the cubic.

T sextic invariant, $= 1 - 20l^3 - 8l^6$,
S quartic „ (Aronhold's) $= -4(l - l^4)$,
R discriminant $= (1 + 8l^3)^3$,
$P = 3h\alpha^2 0 \quad = \{-3l^2\alpha^2 + (1 + 2l^3)\beta\gamma\}x +$ etc.,
$Q = f\alpha^2 0 \quad = (\alpha^2 + 2l\beta\gamma)x +$ etc.,
$B = f\mathrm{a}^2 0 \quad = (\mathrm{a}^2 + 2l\mathrm{bc})x +$ etc.,
$[\mathrm{a}, \mathrm{b}, \mathrm{c} = \alpha(\beta^3 - \gamma^3),\ \beta(\gamma^3 - \alpha^3),\ \gamma(\alpha^3 - \beta^3),]$
$C = f\mathrm{a}0^2 \quad = \mathrm{a}(x^2 + 2lyz) +$ etc.,
$D = f0^3 \quad = x^3 + y^3 + z^3 + 6lxyz$.

Then we have $\quad 2TQ^4 + 6SPQ^3 + 8P^3Q = -R^{\frac{2}{3}}(6C^2 - 8BD)$,

viz. this equation, where each side is a quartic function $(x, y, z)^4$, is an identity when (α, β, γ) are connected by the equation, $f\alpha^3, = \alpha^3 + \beta^3 + \gamma^3 + 6l\alpha\beta\gamma, = 0$; and further

$$QdP - PdQ = -R^{\frac{1}{3}}\{\mathrm{a}(ydz - zdy) + \mathrm{b}(zdx - xdz) + \mathrm{c}(xdy - ydx)\}.$$

Hence writing $\quad \lambda = \dfrac{6h\alpha^2 0}{f\alpha^2 0}, = \dfrac{2P}{Q}, = \dfrac{2(\{-3l^2\alpha^2 + (1 + 2l^3)\beta\gamma\}x + \text{etc.})}{(\alpha^2 + 2l\beta\gamma)x + \text{etc.}}$,

we have $\quad Q^4(\lambda^3 - 3S\lambda - 2T) = R^{\frac{2}{3}}(6C^2 - 8BD)$,

and $\quad Q^2 d\lambda, = 2(QdP - PdQ) = -2R^{\frac{1}{3}}\{\mathrm{a}(ydz - zdy) + \text{etc.}\}$.

112. Supposing now that (x, y, z) are the coordinates of a point on the cubic, then $D = 0$; and taking the square root of each side of the first equation, we may write

$$Q^2\sqrt{\lambda^3 - 3S\lambda - 2T} = -R^{\frac{1}{3}}\sqrt{6}\,.\,C,$$

$$Q^2 d\lambda = -2R^{\frac{1}{3}}\{\mathrm{a}(ydz - zdy) + \text{etc.}\}.$$

We have $\quad d\omega = \dfrac{ydz - zdy}{x^2 + 2lyz} = \dfrac{zdx - xdz}{y^2 + 2lzx} = \dfrac{xdy - ydx}{z^2 + 2lxy}$,

whence $\quad d\omega = \dfrac{\mathrm{a}(ydz - zdy) + \text{etc.}}{C}$;

and we consequently have $\quad \dfrac{d\lambda}{\sqrt{\lambda^3 - 3S\lambda - 2T}} = \dfrac{2}{\sqrt{6}}$,

or as this may also be written

$$\frac{d\lambda}{\sqrt{4\lambda^3 - 12S\lambda - 8T}} = \frac{1}{\sqrt{6}}\,d\omega,$$

which, if $12S$, $8T$ are put $= g_2, g_3$ respectively, takes the Weierstrassian form

$$\frac{d\lambda}{\sqrt{4\lambda^3 - g_2\lambda - g_3}} = \frac{1}{\sqrt{6}}\,d\omega.$$

The conclusion is that for the cubic curve, taking λ a quotient of two linear functions of (x, y, z), the differential $d\omega$ is transformed into $d\lambda \div$ square root of a cubic function of λ: viz. we have thus a form of differential, not the same, but such as that which belongs to the ordinary theory of elliptic functions, and which has been adopted by Weierstrass as a canonical form.

113. The transformation depends on the arbitrary point (α, β, γ) of the cubic: the point (a, b, c) is the tangential of this point, viz. the point of intersection of the tangent at (α, β, γ) with the cubic: we can from (a, b, c) draw four tangents to the cubic, viz. the tangent at (α, β, γ) and three other tangents: the equations of the four tangents being $\frac{2P}{Q}, = \frac{6h\alpha^2 0}{f\alpha^2 0}, = \infty, \lambda_1, \lambda_2, \lambda_3$ respectively; where $\lambda_1, \lambda_2, \lambda_3$ are the roots of the equation $\lambda^3 - 3S\lambda - 2T = 0$.

Suppose for a moment that (α, β, γ) is a point not on the cubic curve, and write $A = \alpha^3 + \beta^3 + \gamma^3 + 6l\alpha\beta\gamma$, we have

$$A^2D^2 + 4AC^3 + 4B^3D - 3B^2C^2 - 6ABCD = 0,$$

for the equation of the six tangents which can be drawn from the point (α, β, γ) to the cubic: when (α, β, γ) is on the cubic $A = 0$, and the equation becomes $B^2(4BD - 3C^2) = 0$, where $B = 0$ is the equation of the tangent at the point (α, β, γ): throwing out the factor B^2 we have $4BD - 3C^2 = 0$, for the equation of the four tangents from (α, β, γ) to the curve; viz. the equation of the four tangents is $2TQ^4 + 6SPQ^3 + 8P^3Q = 0$, or as this may be written

$$Q(2P - \lambda_1 Q)(2P - \lambda_2 Q)(2P - \lambda_3 Q) = 0,$$

viz. the equations of the four tangents are as is mentioned above; it was in fact by these geometrical considerations that Aronhold obtained his results.

114. The foregoing expression for $QdP - PdQ$, say

$$QdP - PdQ = (1 + 8l^3)\{\mathrm{a}(ydz - zdy) + \mathrm{b}(zdx - xdz) + \mathrm{c}(xdy - ydx)\},$$

may be verified without difficulty. Writing for a moment

$$\begin{aligned} QdP - PdQ = {} & (Ax + By + Cz)(Ldx + Mdy + Ndz) \\ & - (Adx + Bdy + Cdz)(Lx + My + Nz), \\ = {} & (BN - CM)(ydz - zdy) - \text{etc.}; \end{aligned}$$

we have

$$\begin{aligned} BN - CM = {} & \{-3l^2\beta^2 + (1 + 2l^3)\gamma\alpha\}(\gamma^2 + 2l\alpha\beta) \\ & -\{-3l^2\gamma^2 + (1 + 2l^3)\alpha\beta\}(\beta^2 + 2l\gamma\alpha) \\ = {} & - 6l^3\alpha\beta^3 + (1 + 2l^3)\alpha\gamma^3 + 6l^3\alpha\gamma^3 - (1 + 2l^3)\alpha\beta^3, \\ = {} & - (1 + 8l^3)\alpha(\beta^3 - \gamma^3) \\ = {} & - (1 + 8l^3)\mathrm{a}, \end{aligned}$$

which proves the theorem.

115. I content myself with a partial verification of the identity

$$2TQ^4 + 6SPQ^3 - 8P^3Q = -(1+8l^3)^2(6C^2 - 8BD);$$

writing herein $x, y, z = 1, -1, 0$, we have $D = 0$, and the equation becomes

$$2TQ^4 + 6SPQ^3 - 8P^3Q + 6(1+8l^3)^2C^2 = 0,$$

where now

$$Q = (\alpha-\beta)(\alpha+\beta-2l\gamma),\ P = (\alpha-\beta)\{-3l^2(\alpha+\beta) - (1+2l^3)\gamma\},$$
$$C = \mathrm{a}+\mathrm{b}-2l\mathrm{c},\ = (\alpha-\beta)\{-\alpha\beta^2 - \alpha^2\beta - \gamma^3 - 2l\gamma(\alpha^2+\alpha\beta+\beta^2)\},$$

which putting therein

$$-\gamma^3 = \alpha^3+\beta^3+6l\alpha\beta\gamma \text{ becomes } = (\alpha-\beta)^2(\alpha+\beta-2l\gamma).$$

Hence writing

$$X = \alpha+\beta-2l\gamma,\ Y = -3l^2(\alpha+\beta) - (1+2l^3)\gamma,$$

we have $\quad Q,\ P,\ C = (\alpha-\beta)X,\ (\alpha-\beta)Y,\ (\alpha-\beta)^2X$:

substituting these values the factor $(\alpha-\beta)^4X$ divides out, and the equation becomes $\quad 2TX^3 + 6SX^2Y - 8Y^3 + 6(1+8l^3)^2(\alpha-\beta)^3X = 0.$

To complete the verification observe that we have $Y + 3l^2X = -(1+8l^3)\gamma$, whence

$$-Y^3 = (1+8l^3)^3\gamma^3 + 9(1+8l^3)^2l^2\gamma^2X + 27(1+8l^3)l^4\gamma X^2 + 27l^6X^3,$$

and herein $-\gamma^3 = \alpha^3+\beta^3+6l\alpha\beta\gamma$, whence

$$-\gamma^3 + \alpha\beta X = (\alpha+\beta)^3 = (X+2l\gamma)^3 = X^3 + 6l\gamma X^2 + 12l^2\gamma^2X + 8l^3\gamma^3,$$

that is $\quad -(1+8l^3)\gamma^3 = X^3 + 6l\gamma X^2 + (12l^2\gamma^2 - 3\alpha\beta)X.$

Hence the equation to be verified becomes

$$2TX^3+6SX^2Y-8\left\{\begin{array}{l}(1+8l^3)^2([X^3+6l\gamma X^2+(12l^2\gamma^2-3\alpha\beta)X]\\ -(1+8l^3)^2 9l^2\gamma^2X^2\\ -(1+8l^3)27l^4\gamma X\\ -27l^6X^3\end{array}\right\}$$
$$+6(1+8l^3)^2(\alpha-\beta)^2X = 0;$$

viz. throwing out the factor X, this is

$$\{2T - 8(1+8l^3)^2 + 216l^6\}X^2 + 6SXY - 48(1+8l^3)^2l\gamma X + 216(1+8l^3)l^4\gamma X$$
$$-(1+8l^3)^2\{96l^2\gamma^2 - 24\alpha\beta - 72l^2\gamma^2 - 6(\alpha-\beta)^2\} = 0,$$

where the last term is $\quad = +6(1+8l^3)^2\{(\alpha+\beta)^2 - 4l^2\gamma^2\}$,

viz. this is $\quad = 6(1+8l^3)^2(\alpha+\beta+2l\gamma)X$,

and there is again the factor X which can be thrown out: the equation thus becomes

$$[2T - 8(1+8l^3)^2 + 216l^6]X + 6SY - 48(1+8l^3)^2l\gamma + 216(1+8l^3)l^4\gamma$$
$$+6(1+8l^3)^2(\alpha+\beta+2l\gamma) = 0.$$

This may be written

$$[2T - 8(1+8l^3)^2 + 216l^6]X + 6S[-3l^2X - (1+8l^3)\gamma] - 48(1+8l^3)^2 l\gamma$$
$$+ 216(1+8l^3)l^4\gamma + 6(1+8l^3)^2(X+4l\gamma) = 0.$$

or finally it is

$$[2T - 8(1+8l^3)^2 + 216l^6 - 18l^2S + 6(1+8l^3)^2]X$$
$$+ [-6l^{-1}S - 48(1+8l^3) + 216l^3 + 24(1+8l^3)](1+8l^3)l\gamma = 0;$$

and substituting for T, S their values $1 - 20l^3 - 8l^6$ and $-4l + 4l^4$ respectively, the coefficients of X and $(1+8l^3)l\gamma$ are separately $= 0$, and the equation is thus verified.

116. The foregoing equation $\lambda = \frac{6h\alpha^2 0}{f\alpha^2 0}$ regarding therein λ as an arbitrary parameter and (x, y, z) as current coordinates is the equation of an arbitrary line through the point (a, b, c) of the cubic: it meets the cubic in two other points depending, of course, on the value of λ; and the coordinates of either of these is thus expressible, irrationally, in terms of λ, the expressions involving the radical $\sqrt{\lambda^3 - 3S\lambda - 2T}$: from the values of x, y, z in terms of λ we should be able to deduce the foregoing equation $\frac{2}{\sqrt{6}}\,d\omega = \frac{d\lambda}{\sqrt{\lambda^3 - 3S\lambda - 2T}}$: the expressions assume a peculiarly simple form when (α, β, γ) instead of being an arbitrary point of the cubic is a point of inflexion of the cubic; and it is easy to see *à priori* why this is so: in fact if we assume $x:y:z = u + \alpha\sqrt{\Lambda} : v + \beta\sqrt{\Lambda} : w + \gamma\sqrt{\Lambda}$, where u, v, w are linear functions and Λ a cubic function of λ; then the locus is a cubic curve, and corresponding to the value $\lambda = \infty$, we have $x:y:z = \alpha:\beta:\gamma$, viz. the curve passes through the point (α, β, γ): moreover, it can be shown that this point is an inflexion of the curve; expressions of the foregoing simple form thus only exist in the case where the point (α, β, γ) is an inflexion, and the formulæ referring to an arbitrary point (α, β, γ) of the curve are necessarily of a more complex form.

117. To work this out we start from the foregoing equation

$$\lambda = \frac{6h\alpha^2 0}{f\alpha^2 0} = \frac{2(\{-3l^2\alpha^2 + (1+2l^3)\beta\gamma\}x + \text{etc.})}{(\alpha^2 + 2l\beta\gamma)x + \text{etc.}}$$

which putting therein $L = \lambda + 6l^2$, $M = l\lambda - (1+2l^3)$, and

$$A, B, C = L\alpha^2 + 2M\beta\gamma,\ L\beta^2 + 2M\gamma\alpha,\ L\gamma^2 + 2M\alpha\beta,$$

becomes $Ax + By + Cz = 0$, the equation of a line through the point (a, b, c), $= (\alpha(\beta^3 - \gamma^3),\ \beta(\gamma^3 - \alpha^3),\ \gamma(\alpha^3 - \beta^3))$ as before: and we have to find the intersections of this line with the cubic $x^3 + y^3 + z^3 + 6lxyz = 0$. We have

$$C^3(x^3 + y^3) - (Ax + By)^3 - 6lC^2(Ax + By)xy = 0:$$

the cubic function contains as we know the factor $\mathrm{b}x - \mathrm{a}y$, and in the remaining quadric factor it is easy to calculate the coefficients of x^2 and y^2: we thus obtain the identity

$$\begin{aligned} & C^3(x^3+y^3) - (Ax+By)^3 - 6lC^2(Ax+By)\,xy \\ &\quad = (\mathrm{b}x - \mathrm{a}y)\{[(-\beta^2 - 6l\gamma\alpha)L^3 + 6\alpha\gamma L^2M - 8\beta^2M^3]\,x^2 \\ &\qquad\qquad + 2H \quad xy \\ &\qquad\qquad + [(-\alpha^2 - 6l\alpha\beta)L^3 + 6\beta\gamma L^2M - 8\alpha^2M^3]\,y^2\}, \end{aligned}$$

from which the as yet unknown coefficient $2H$ is to be obtained. This is most easily effected by assuming $x, y = \alpha, -\beta$; values which give

$$x^3 + y^3 = \alpha^3 - \beta^3,\ Ax + By = L(\alpha^3 - \beta^3),\ \mathrm{b}x - \mathrm{a}y = -\alpha\beta(\alpha^3 - \beta^3):$$

the whole equation becomes divisible by $\alpha^3 - \beta^3$ and omitting this factor we have

$$\begin{aligned} & C^3 - L^3(\alpha^3 - \beta^3)^2 + 6lC^2L\alpha\beta \\ &\quad = \alpha\beta\{[2\alpha^2\beta^2 + 6l\gamma(\alpha^3+\beta^3)]L^3 - 6\gamma(\alpha^3+\beta^3)L^2M + 16\alpha^2\beta^2M^3\} + 2H\alpha^2\beta^2, \end{aligned}$$

where for C is to be substituted its value $L\gamma^2 + 2M\alpha\beta$, and we may also reduce by $\alpha^3 + \beta^3 + \gamma^3 + 6l\alpha\beta\gamma = 0$. The left-hand side is

$$C^3 - L^3(\alpha^3 + \beta^3)^2 - 4\alpha^3\beta^3L^3 + 6lC^2L\alpha\beta,$$

which after reduction is found to contain the factor $\alpha\beta$; and omitting this factor and reducing also the right-hand side the whole equation becomes

$$\begin{aligned} & L^3(-6l\gamma^4 - 36l^2\gamma^2\alpha\beta + 4\alpha^2\beta^2) && = L^3(-6l\gamma^4 - 36l^2\gamma^2\alpha\beta + 2\alpha^2\beta^2) \\ & + L^2M(6\gamma^4 + 24l\gamma^2\alpha\beta) && + L^2M(6\gamma^4 + 36l\gamma^2\alpha\beta) \\ & + LM^2(12\gamma^2\alpha\beta + 24l\alpha^2\beta^2) && \\ & + M^3(8\alpha^2\beta^2) && + M^3(16\alpha^2\beta^2) \\ & && + 2H\alpha\beta; \end{aligned}$$

omitting here the terms which destroy each other, the equation again divides by $\alpha\beta$, and we thus obtain the value of H; and the required identity is

$$\begin{aligned} & C^3(x^3+y^3) - (Ax+By)^3 - 6lC^2(Ax+By)\,xy \\ &\quad = (\mathrm{b}x - \mathrm{a}y)\{[(-\beta^2 - 6l\gamma\alpha)L^3 + 6\alpha\gamma L^2M - 8\beta^2M^3]\,x^2 \\ &\qquad\qquad + [\alpha\beta L^3 - 6l\gamma^2L^2M + (6\gamma^2 + 12l\alpha\beta)LM^2 - 4\alpha\beta M^3]\,2xy \\ &\qquad\qquad + [(-\alpha^2 - 6l\beta\gamma)L^3 + 6\beta\gamma L^2M - 8\alpha^2M^3]\,y^2\}. \end{aligned}$$

Hence putting for shortness

$$\begin{aligned} \mathfrak{A} &= (\alpha^2 + 6l\beta\gamma)L^3 - 6\beta\gamma L^2M && + 8\alpha^2M^3, \\ \mathfrak{B} &= (\beta^2 + 6l\gamma\alpha)L^3 - 6\alpha\gamma L^2M && + 8\beta^2M^3, \\ \mathfrak{H} &= \alpha\beta L^3 - 6l\gamma^2L^2M + (6\gamma^2 + 12l\alpha\beta)LM^2 - 4\alpha\beta M^3, && \end{aligned}$$

the equation in (x, y) is

$$\mathfrak{B}x^2 - 2\mathfrak{H}xy + \mathfrak{A}y^2 = 0,$$

giving

$$\mathfrak{B}x - \mathfrak{H}y = \sqrt{\mathfrak{H}^2 - \mathfrak{A}\mathfrak{B}}\,y,$$

or say

$$x : y = \mathfrak{H} + \sqrt{\mathfrak{H}^2 - \mathfrak{A}\mathfrak{B}} : \mathfrak{B}.$$

We find without difficulty, reducing always by $\alpha^3+\beta^3+\gamma^3+6l\alpha\beta\gamma=0$,

$$\begin{array}{rllll}
\tfrac{1}{6}(\mathfrak{H}^2-\mathfrak{A}\mathfrak{B}) = & l\gamma^4 & & & L^6 \\
+(-\gamma^4 & +\ 4l\alpha\beta\gamma^2) & & & L^5M^2 \\
+(\ \ 6l^2\gamma^4 & -\ 4\alpha\beta\gamma^2 & +\ 4l\alpha^2\beta^2) & & L^4M^2 \\
+(-4l\gamma^4 & +\ 24l^2\alpha\beta\gamma^2 & -\ 4\alpha^2\beta^2) & & L^3M^3 \\
+(-2\gamma^4 & -\ 16l\alpha\beta\gamma^2 & +\ 24l\alpha^2\beta^2) & & L^2M^4 \\
+(& -\ 8\alpha\beta\gamma^2 & -\ 16l\alpha^2\beta^2) & & LM^5 \\
+(& & -\ 8\alpha^2\beta^2) & & M^6,
\end{array}$$

which is $\quad = (lL-M)(L^3+6lLM^2+2M^3)(\gamma^2L+2\alpha\beta M)^2.$

But we have $\quad lL-M=1+8l^3,$

$$L^3+6lLM^2+2M^3=(1+8l^3)(\lambda^3-3S\lambda-2T),$$

and the equation thus is

$$\tfrac{1}{6}(\mathfrak{H}^2-\mathfrak{A}\mathfrak{B})=(1+8l^3)^2(\lambda^3-3S\lambda-2T)\left((\gamma^2+2l\alpha\beta)\lambda+\{6l^2\gamma^2-2(1+2l^3)\alpha\beta\}\right)^2,$$

showing that the solution involves the radical $\sqrt{\lambda^3-3S\lambda-2T}$.

118. If (α, β, γ) is the inflexion $(1, -1, 0)$, the expression for λ is here

$$\lambda=\frac{-6l^2x-6l^2y-2(1+2l^3)z}{x+y-2lz};$$

the equation in (x, y) is

$$(L^3+8M^3)x^2+(2L^3+24lLM^2-8M^3)xy+(L^3+8M^3)y^2=0,$$

or as this may be written

$$(L^3+6lLM^2+2M^3)(x+y)^2+(-6lLM^2+6M^3)(x-y)^2=0,$$

say $\quad (L^3+6lLM^2+2M^3)(x+y)^2=6M^2(lL-M)(x-y)^2,$

viz. we thus have $\quad \sqrt{\lambda^3-3S\lambda-2T}\,(x+y)=M\sqrt{6}(x-y),$

or substituting for M its value

$$x=\sqrt{6}\{l\lambda-(1+2l^3)\}+\sqrt{\lambda^3-3S\lambda-2T},$$
$$y=\sqrt{6}\{l\lambda-(1+2l^3)\}-\sqrt{\lambda^3-3S\lambda+2T},$$

whence also $\quad z=\sqrt{6}(\lambda+6l^2),$

these values satisfying identically $(\lambda+6l^2)(x+y)-2[l\lambda-(1+2l^3)]z=0$, and $x^3+y^3+z^3+6lxyz=0$:

119. Starting from these values we in fact easily obtain

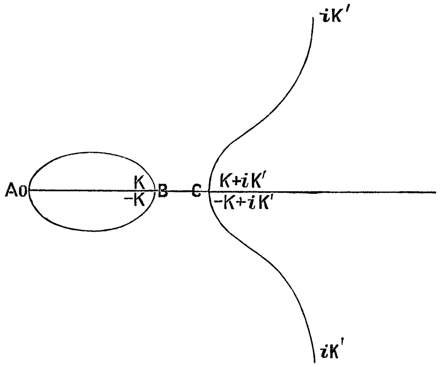
$$xdy-ydx=\frac{-\sqrt{6}\,d\lambda}{\sqrt{\lambda^3-3S\lambda-2T}}\times$$
$$\{l\lambda^3+(-3-6l^3)\lambda^2+(-12l^2+12l^5)\lambda+(-8l-92l^4-8l^7)\},$$
$$z^2+2lxy=-2\{\text{Do.}\}.$$

and hence $\quad d\omega=\dfrac{xdy-ydx}{z^2+2lxy}=\dfrac{\frac{1}{2}\sqrt{6}\,d\lambda}{\sqrt{\lambda^3-3S\lambda-2T}}.$

The same result might of course have been obtained from the values of x, z or y, z, the factor which divides out being in each of these cases irrational.

The Cubic $y^2 = x(1-x)(1-k^2x)$. Art. Nos. 120 to 130 (*several sub-headings*).

120. The curve is a semi-cubical parabola, symmetrical in regard to the axis of x; and if as usual k^2 is taken to be real, positive and less than 1, then the curve consists of an oval, and an infinite portion which may be called the anguis. See Figure.



The equation is satisfied by

$$x = \operatorname{sn}^2 u,$$

$$y = \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u.$$

Observe that the periods for these combinations of the elliptic functions are $2K$, $2iK'$; in fact

$$\begin{array}{llll} \operatorname{sn}(u+2K) & = -\operatorname{sn} u, & \operatorname{sn}(u+2iK') & = \operatorname{sn} u, \\ \operatorname{cn} \quad `` & = -\operatorname{cn} u, & \operatorname{cn} \quad `` & = -\operatorname{cn} u, \\ \operatorname{dn} \quad `` & = \phantom{-}\operatorname{dn} u, & \operatorname{dn} \quad `` & = -\operatorname{dn} u, \end{array}$$

whence the sn^2, and sn.cn.dn are each unaltered by the change of u into $u+2K$ or $u+2iK'$. Hence to a given point (x, y) on the curve, the argument u is not $=$ a determinate value u_0, for it may be equally well taken to be $= u_0 + 2mK + 2m'iK'$, where m, m' are any positive or negative integers whatever: we express this by $u \equiv u_0$, or say u is congruent to u_0. But when u is thus given by a congruence $u \equiv u_0$ where u_0 has a determinate value, the point on the curve is uniquely determined. It is, however, to be noticed that a congruence $2u \equiv u_0$ does not uniquely determine the point on the curve: there are in fact four incongruent values of u, viz. $\tfrac{1}{2}u_0$, $\tfrac{1}{2}u_0 + K$, $\tfrac{1}{2}u_0 + iK'$, $\tfrac{1}{2}u_0 + K + iK'$, and the point on the curve is thus one of the four points belonging to these values of u respectively.

121. If to fix the ideas we select for each point of the curve one of the congruent values of the argument, we may assume for the oval, u real: at A, $u = 0$; from A to B above the axis u positive and at B, $= K$; below the axis u negative and at B, $= -K$; there is thus a discontinuity, K, $-K$ at B, but the two values are congruent. For the anguis, $u = iK' +$ real value v: above the axis v positive, viz. at infinity $v = 0$, and at C, $= K$; below the axis v negative,

viz. at infinity $v = 0$ and at C, $= -K$: there thus is a discontinuity $iK' + K$, $iK' - K$ at C, but the two values are congruent. Observe that for points opposite to each other in regard to the axis, the arguments are, for points on the oval $u, -u$; for points on the anguis $iK' + v$, $iK' - v$: but that we have $iK' - v \equiv -(iK' - v)$.

122. The pure theorem gives for three points u_1, u_2, u_3 in a line

$$du_1 + du_2 + du_3 = 0\,;$$

and thence $u_1 + u_2 + u_3 = C$. The constant C cannot have a determinate value (for if it had, then assuming the values of u_1 and u_2 at pleasure u_3 would have the determinate value $= C - u_1 - u_2$), but must be given by a congruence: or what is the same thing, assigning to C any admissible value, we must instead of $u_1 + u_2 + u_3 = C$, write $u_1 + u_2 + u_3 \equiv C$. Taking any particular line, for instance the tangent at A, we have $u_1, u_2, u_3 = 0, 0, iK'$; whence $C = iK'$; and we have $u_0 + u_2 + u_3 \equiv iK'$, viz. this is the relation between the arguments u_1, u_2, u_3 belonging to the points of intersection of the cubic with a line: in particular for a line at right angles to the axis we have $u_1, u_2, u_3 = \alpha, -\alpha, iK'$ or $= iK' + \beta, iK' - \beta, iK'$ (according as the line cuts the oval or the anguis) and the congruence is in each case satisfied. But I shall in general instead of $\equiv$ use the sign $=$, understanding it as in general meaning $\equiv$, and only replacing it by this sign when for clearness it seems necessary to do so.

Writing $\operatorname{sn} u_1, \operatorname{cn} u_1, \operatorname{dn} u_1 = s_1, c_1, d_1$, and so in other cases, the condition in order that the three points may be in a line is

$$\begin{vmatrix} s_1^2, & s_1c_1d_1, & 1 \\ s_2^2, & s_2c_2d_2, & 1 \\ s_3^2, & s_3c_3d_3, & 1 \end{vmatrix} = 0,$$

a relation which must be satisfied when the arguments are connected by the foregoing relation $u_1 + u_2 + u_3 \equiv iK'$.

We can show from this equation alone that s_3^2 and $s_3c_3d_3$ are expressible rationally in terms of $s_1^2, s_1c_1d_1, s_2^2, s_2c_2d_2$; in fact, writing therein x_1, y_1 in place of $s_1^2, s_1c_1d_1$, etc., we thence have $x_3, y_3, 1 = \lambda x_1 + \mu x_2, \lambda y_1 + \mu y_2, \lambda + \mu$, and substituting in $y_3^2 = x_3(1 - x_3)(1 - k^2x_3)$, we obtain an equation $\lambda\mu(F\lambda + G\mu) = 0$, that is $F\lambda + G\mu = 0$, or say $\lambda, \mu = G, -F$, and thence

$$x_3 = s_3^2 = \frac{Gx_1 - Fx_2}{G - F}, \quad y_3 = s_3c_3d_3 = \frac{Gy_1 - Gy_2}{G - F}\,;$$

the values of F, G are easily found to be

$$F = y_1^2 + 2y_1y_2 - x_1(1-x_1)(1-k^2x_2) - x_1(1-k^2x_1)(1-x_2) - (1-x_1)(1-k^2x_1)x_2,$$
$$G = 2y_1y_2 + y_2^2 - x_2(1-x_2)(1-k^2x_1) - x_2(1-k^2x_2)(1-x_1) - (1-x_2)(1-k^2x_2)x_1,$$

or as these may also be written

$$F = -(y_1-y_2)^2 + (x_1-x_2)^2(1+k^2+k^2(x_1+x_2)) + k^2(x_1-x_2)^2x_1,$$
$$G = -(y_1-y_2)^2 + (x_1-x_2)^2(1+k^2+k^2(x_1+x_2)) + k^2(x_1-x_2)^2x_2,$$

where of course x_1, y_1, x_2, y_2 should be replaced by their values s_1^2, $s_1c_1d_1$, s_2^2, $s_2c_2d_2$. This is in fact the ordinary process of finding the third point of intersection of a cubic by a line which meets it in two given points.

Writing $iK'-u_3 = u$, and s, c, d for the sn, cn, dn of u, we have $s_3, c_3, d_3 = -\frac{1}{ks}, \frac{id}{ks}, \frac{ic}{s}$, whence $s_3^2 = \frac{1}{k^2s^2}$, $s_3c_3d_3 = \frac{cd}{k^2s^3}$, and the determinant equation becomes

$$\begin{vmatrix} s_1^2, & s_1c_1d_1, & 1 \\ s_2^2, & s_2c_2d_2, & 1 \\ 1, & \frac{cd}{s}, & k^2s^2 \end{vmatrix}$$

that is
$$(1-k^2s^2s_2^2)s_1c_1d_1 - (1-k^2s^2s_1^2)s_2c_2d_2 - (s_1^2-s_2^2)\frac{cd}{s} = 0,$$

corresponding to the relation $u = u_1 + u_2$ of the arguments. This is easily verified: we have

$$s_1 = \frac{sc_2d_2 - s_2cd}{1-k^2s^2s_2^2}, \quad s_2 = \frac{sc_1d_1 - s_1cd}{1-k^2s^2s_1^2}, \quad s = \frac{s_1^2-s_2^2}{s_1c_2d_2 - s_2c_1d_1};$$

the equation thus becomes

$$(sc_2d_2 - s_2cd)c_1d_1 - (sc_1d_1 - s_1cd)c_2d_2 - (s_1^2-s_2^2)\frac{cd}{s} = 0.$$

that is
$$cd\left\{-s_2c_1d_1 + s_1c_2d_2 - \frac{s_1^2-s_2^2}{s}\right\} = 0,$$

which is right.

The Four Tangents from a Point of the Cubic.

123. Suppose that the line is a tangent to the cubic, say the line touches the cubic at the point u, and again meets it at the point w: then instead of u_1, u_2, u_3 we have u, u, w: and the relation becomes $2u + w \equiv iK'$.

Here u being given, w is uniquely determined: viz. given the argument u of the point of contact, we have a unique value for the argument w of the tangential. But given w, we have $2u = iK' - w$, and we have thus for u the four values $\frac{1}{2}(iK'-w)$, Do. $+K$, Do. $+iK'$, Do. $+K+iK'$, corresponding to the four tangents which can be drawn from the point w to the cubic.

The tangents are real for a point of the anguis, and for such a point we may write $w = iK' + v$, where v is real and included between the values $\pm K$; the corresponding values of u are

$$u_1 = -\tfrac{1}{2}v,\ u_2 = -\tfrac{1}{2}v + K,\ u_3 = -\tfrac{1}{2}v + iK',\ u_4 = -\tfrac{1}{2}v + K + iK':$$

the first and second of these belong to tangents to the oval, the third and fourth to tangents to the anguis. We may further distinguish a tangent according as it passes between or does not pass between the vertices B and C: say in the former case it is intermediate, and in the latter case extramediate: and we then see that for the tangents from point $iK' + v$ of anguis,

$u = -\tfrac{1}{2}v$ for intermediate to oval,
$u = -\tfrac{1}{2}v + K$ " extramediate to oval,
$u = -\tfrac{1}{2}v + iK'$ " extramediate to anguis,
$u = -\tfrac{1}{2}v + K + iK'$ " intermediate to anguis.

124. We may make a corresponding division of the real lines which meet the curve in three real points: any such line meets the oval twice (and then of course the anguis once), or else it meets the anguis three times: and taking the arguments to be u_1, u_2, u_3 we have

$\tfrac{1}{2}(u_1 + u_2 + u_3) = \tfrac{1}{2}iK'$ for intermediate line meeting oval twice,
" $= \tfrac{1}{2}iK' + K$ " extramediate line, Do.,
" $= -\tfrac{1}{2}iK'$ " extramediate line meeting anguis three times,
" $= -\tfrac{1}{2}iK' + K$ " intermediate line, Do.

125. Returning to the tangents, the point $iK' + v$ may be an inflexion: we have then the point of contact of the intermediate tangent to the anguis coinciding with the point $iK' + v$; viz. $iK' + v \equiv -\tfrac{1}{2}v + K + iK'$, or say $= -\tfrac{1}{2}v \pm K + iK'$: that is $v = \pm\tfrac{2}{3}K$; or $iK' + \tfrac{2}{3}K$ and $iK' - \tfrac{2}{3}K$ are the arguments for the real points of inflexion, above and below the axis respectively.

126. Write for a moment the equation in the form $y^2 = Bx + Cx^2 + Dx^3$, then if (α, β) be a point on the curve $(\beta^2 = B\alpha + C\alpha^2 + D\alpha^3)$, and we consider the intersections of the curve with the line $y - \beta = m(x - \alpha)$ we find for the remaining two intersections

$$B + C(x + \alpha) + D(x^2 + \alpha x + \alpha^2) = 2m\beta + m^2(x - \alpha).$$

If the line be a tangent, this will be satisfied by $x - \alpha$; the condition for this is $2m\beta = B + 2C\alpha + 3D\alpha^2$, and supposing this satisfied, then throwing out the factor $x - \alpha$ we obtain $C + D(x + 2\alpha) = m^2$, giving $Dx = m^2 - C - 2D\alpha$ for the coordinate x of the tangential of the point (α, β).

In the case of an inflexion $x = \alpha$, and we have

$$m^2 = C + 3D\alpha, = \frac{(B + 2C\alpha + 3D\alpha^2)^2}{4(B\alpha + C\alpha^2 + D\alpha^3)},$$

giving for α the equation

$$3D^2\alpha^4 + 4CD\alpha^3 + 6BD\alpha - B^2 = 0,$$

or for B, C, D writing their values 1, $-(1 + k^2)$, k^2 this is

$$3k^4\alpha^4 - 4k^2(1 + k^2)\alpha^3 + 6k^2\alpha^2 - 1 = 0$$

for the x-coordinate of the inflexion. There is one negative root, and one or three positive roots; but only one positive root giving a real value of β, and corresponding hereto we have the two real inflexions on the anguis.

It should be possible, from the formulæ of No. 122, writing therein $(x_2, y_2) = (x_1, y_1)$ to arrive at the foregoing result, say $Dx_3 = m^2 - C - 2Dx_1$, but the functions F and G present themselves in vanishing forms, and the reduction is not immediate.

127. The general condition for an inflexion is $3u \equiv iK'$: the nine inflexions thus are $u = iK'$, inflexion at infinity, $u = iK' \pm \frac{2}{3}K$, the two real inflexions, and besides $u = \pm \frac{1}{3}iK'$, $u = \pm \frac{1}{3}iK' \pm \frac{2}{3}K$, six imaginary inflexions.

The Sextactic Points.

128. The vertices A, B, C are each of them a sextactic point: in fact writing the equation in the form $y^2 = x - (1 + k^2)x^2 + k^2x^3$, we see at once that the conic $y^2 = x - (1 + k^2)x^2$ meets the curve in the point A counting six times: and there is obviously a like proof for the vertices B and C respectively. Hence for the six intersections with any conic whatever we have the condition

$$u_1 + u_2 + u_3 + u_4 + u_5 + u_6 \equiv 0:$$

and for the sextactic points we have the condition $6u \equiv 0$. This gives the 36 points $u = \frac{1}{6}(2mK + 2m'iK')$ or say $= \frac{1}{3}(mK + m'iK')$, $m = 0, \pm 1, \pm 2, 3$, $m' = 0, \pm 1, \pm 2, 3$: but among these are included the 9 inflexions (each of these being an improper sextactic point, the conic becoming the tangent taken twice) and there remain 27 points: among these are included the three vertices ($u = 0$, K, $iK' + K$) points of contact with the tangents from the inflexion at infinity; and of the remaining 24 points 6 are real, viz. these are the points $u = \pm \frac{1}{3}K$, $\pm \frac{2}{3}K$ on the oval, and the points $iK' \pm \frac{1}{3}K$ on the anguis: these are in fact the points of contact of the tangents from the real inflexions, viz. the three tangents from the inflexion $iK' + \frac{2}{3}K$ touch the oval in the points $\frac{2}{3}K$, $-\frac{1}{3}K$, and the anguis in the point $iK' - \frac{1}{3}K$; the three tangents from the inflexion $iK' - \tfrac{2}{3}K$ touch the oval in the points $-\tfrac{2}{3}K$, $\tfrac{1}{3}K$, and the anguis in the point $iK' + \tfrac{1}{3}K$.

Formulæ Relating to the Tangents from the Vertices.

129. I annex some formulæ relating to the tangents to the curve from the vertices A, B, C respectively. We have from each vertex four tangents say $\rho = 0$, $\sigma = 0$, symmetrically situate in regard to the axis, and $\rho' = 0$, $\sigma' = 0$, symmetrically situate in regard to the axis: the line joining the points of contact of ρ, σ is a line $\tau = 0$ at right angles to the axis, and that joining the points of contact of ρ', σ' is a line $\tau' = 0$ at right angles to the axis.

VERTEX A, TANGENTS IMAGINARY. COORDINATES OF POINT OF CONTACT.

$$\rho,\ \sigma,\ \tau = y - i(1+k)x,\ y + i(1+k)x,\quad kx + 1;\qquad x = -\frac{1}{k},\ y = \mp\frac{i(1+k)}{k}$$

$$\rho',\ \sigma',\ \tau' = y - i(1-k)x,\ y + i(1-k)x,\ -kx + 1;\qquad x = \frac{1}{k},\ y = \pm\frac{i(1-k)}{k},$$

equation of curve is, $f, = y^2 - x(1-x)(1-k^2x), = \rho\sigma - x\tau^2, = \rho'\sigma' - x\tau'^2, = 0.$

VERTEX B, TANGENTS IMAGINARY. CONTACTS.

$$\rho',\ \sigma',\ \tau' = y - (k - ik')(1-x),\ y + (k - ik')(1-x),\ kx - (k - ik');\ x = 1 - \frac{ik'}{k},\ y = \mp\frac{ik'}{k}(k + ik')$$

$$\rho',\ \sigma',\ \tau' = y - (k - ik')(1-x),\ y + (k - ik')(1-x),\ kx - (k + ik');\ x = 1 + \frac{ik'}{k},\ y = \pm\frac{ik'}{k}(k - ik'),$$

equation is $f = \rho\sigma + (1-x)\tau^2, = \rho'\sigma' + (1-x)\tau'^2, = 0,$

VERTEX C, TANGENTS REAL. CONTACTS.

$$\rho,\ \sigma,\ \tau = y - \frac{1}{1+k'}(1-k^2x),\ y + \frac{1}{1+k'}(1-k^2x),\ x - \frac{1}{1+k'};\quad x = \frac{1}{1+k'},\ y = \frac{\pm k'}{1+k'}$$

$$\rho',\ \sigma',\ \tau' = y - \frac{1}{1-k'}(1-k^2x),\ y + \frac{1}{1-k'}(1-k^2x),\ x - \frac{1}{1-k'};\quad x = \frac{1}{1-k'},\ y = \frac{\pm k'}{1-k'},$$

equation is $f, = \rho\sigma - (1-k^2x)\tau, = \rho'\sigma' - (1-k^2x)\tau', = 0.$

130. These linear functions ρ, σ, etc. considering therein x, y as denoting $\operatorname{sn}^2 u$, $\operatorname{sn} u \operatorname{cn} u \operatorname{dn} u$ respectively present themselves as the numerators and denominators of some formulæ given No. 105 of my Elliptic Functions (1876), see p. 76: viz. we have

$$\operatorname{sn}^2(u + \tfrac{1}{2}K) = \frac{1}{1+k'}\,\frac{1 - k^2x + (1+k')y}{1 - k^2x + (1-k')y}$$

which is

$$= \frac{1}{1-k'}\,\frac{y + \dfrac{1}{1+k'}(1-k^2x)}{y + \dfrac{1}{1+k'}(1-k^2x)},\ = \frac{1}{1-k'}\,\frac{\sigma}{\sigma'},\ \text{(vertex C)}.$$

$$\operatorname{sn}^2(u + \tfrac{1}{2}iK') = \frac{1}{k}\,\frac{(1+k)x + iy}{(1+k)x - iy}$$

which is $$= -\frac{1}{k}\frac{y - i(1+k)x}{y + i(1+k)x}, \qquad = -\frac{1}{k}\frac{\rho}{\sigma}, \text{ (vertex A).}$$

and

$$\mathrm{sn}^2(u + \tfrac{1}{2}K + \tfrac{1}{2}iK') = \frac{k + ik'}{k}\frac{1 - x + (k - ik)y}{1 - x + (k + ik)y}$$

which is $$= \frac{k - ik'}{k}\frac{y + (k + ik')(1 - x)}{y + (k - ik')(1 - x)}, \qquad = \frac{k - ik'}{k}\frac{\sigma}{\sigma'}, \text{ (vertex B).}$$

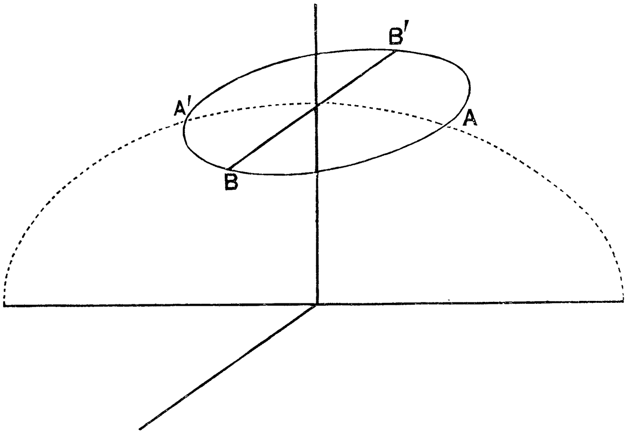
Observe here that in the second formula we have a pair of tangents ρ, σ which belong to a chord τ through the inflexion at ∞; but in the first and third formulæ we have tangents σ, σ' not forming such a pair. This is as it should be, for the zero and infinity of $\mathrm{sn}^2(u + \frac{1}{2}iK')$ are $u = -\frac{1}{2}iK'$, $u = \frac{1}{2}iK'$ which belong to points in lineâ with the inflexion at infinity: but for $\mathrm{sn}^2(u + \frac{1}{2}K)$ the zero is $u = -\frac{1}{2}K$, and the infinity is $u = iK' - \frac{1}{2}K$, which do not belong to points in lineâ with the inflexion at infinity: and the like for $\mathrm{sn}^2(u + \frac{1}{2}K + \frac{1}{2}iK')$.

Fixed Curve a Quartic in Space, the Quadri-quadric Curve
$y^2 = 1 - x^2$, $z^2 = 1 - k^2x^2$. Art. Nos. 131 to 135.

131. It is assumed that k^2 is real, positive, and less than unity: the curve may be regarded as the intersection of the two cylinders

$$x^2 + y^2 = 1, \ k^2x^2 + z^2 = 1;$$

but there is through it a third cylinder $y^2 - k^2z^2 = k'^2$. The cylinder $k^2x^2 + z^2 = 1$, or say the horizontal cylinder has for its section an ellipse axes $\dfrac{1}{k}$ and 1 respectively: and it is pierced by the cylinder $x^2 + y^2 = 1$, or say the vertical cylinder, in two detached ovals (of double curvature) lying on opposite sides of the plane of xy: only the upper oval $ABA'B'$ is shown in the figure.



Each of the vertices A, A', B, B' is an inflexion,* viz. a point such that the osculating plane at that point meets the curve in the point counting four times. We may consider two generating lines of the horizontal cylinder, each meeting the oval in two points; the plane through the generating lines meets the curve in these four points, and when the generating lines come each of them to coincide with the tangent at A, we have the osculating plane meeting the curve in the point A counting four times. The like reasoning, with two generating lines of the third cylinder, shows that the vertex B is an inflexion.

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* There are in all 16 inflexions, 4 in each of the planes $x = 0$, $y = 0$, $z = 0$, and 4 in the plane infinity.

132. The equations are satisfied by writing therein $x, y, z = \operatorname{sn} u, \operatorname{cn} u, \operatorname{dn} u$: the periods are here $4K$, $4iK'$: hence to a given point on the curve the argument is not $u =$ a determinate value u_0, but it may equally well be taken to be $= u_0 + 4mK + 4m'iK'$, where m and m' are any positive or negative integers: we express this by $u \equiv u_0$, or say u congruent to u_0. For the upper oval u may be taken to be real, and to be $= 0$ at B, positive for the half oval BAB', and negative for the half oval $BA'B'$; having the values $+K$, $-K$ at A and A' respectively, and the discontinuity $2K$, $-2K$ at B', these two values being congruent. For the lower oval we have $u = 2iK' +$ real value v.

For the intersections of the curve with a plane we have

$$du_1 + du_2 + du_3 + du_4 = 0\text{; whence } u_1 + u_2 + u_3 + u_4 = C;$$

and by taking the plane to be the osculating plane at B, we find 0 as a value of the constant, and the relation thus is $u_1 + u_2 + u_3 + u_4 \equiv 0$. Writing as before $\operatorname{sn} u_1, \operatorname{cn} u_1, \operatorname{dn} u_1, = s_1, c_1, d_1$ and so in other cases, the relation between the elliptic functions is

$$\begin{vmatrix} s_1, & c_1, & d_1, & 1 \\ s_2, & c_2, & d_2, & 1 \\ s_3, & c_3, & d_3, & 1 \\ s_4, & c_4, & d_4, & 1 \end{vmatrix} = 0;$$

It is important to remark that giving three of the points the fourth point is determined uniquely: that is the equation really gives s_4, c_4, d_4 each as a rational function of the $s_1, c_1, d_1, s_2, c_2, d_2, s_3, c_3, d_3$.

In fact we may write $s_4 = \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3$, and similarly for c_4 and d_4, and $1 = \lambda_1 + \lambda_2 + \lambda_3$: substituting in $s_4^2 + c_4^2 - 1 = 0$, $k^2 s_4^2 + d_4^2 - 1 = 0$, we have

$$\begin{aligned} X_{23}\lambda_2\lambda_3 + X_{31}\lambda_3\lambda_1 + X_{12}\lambda_1\lambda_2 &= 0, \\ Y_{23} \;\text{,,}\; + Y_{31} \;\text{,,}\; + Y_{12} \;\text{,,}\; &= 0, \end{aligned}$$

where $X_{12} = s_1 s_2 + c_1 c_2 - 1$, $Y_{12} = k^2 s_1 s_2 + d_1 d_2 - 1$, etc.; we thence have

$$\begin{aligned} \lambda_2\lambda_3 : \lambda_3\lambda_1 : \lambda_1\lambda_2 &= X_{31}Y_{12} - X_{12}Y_{31} : X_{12}Y_{23} - X_{23}Y_{12} : X_{23}Y_{31} - X_{31}Y_{23} \\ &= \qquad A_1 \qquad : \qquad A_2 \qquad : \qquad A_3\text{, suppose;} \end{aligned}$$

that is $\lambda_1 : \lambda_2 : \lambda_3 = A_2 A_3 : A_3 A_1 : A_1 A_2$.

and consequently

$$\begin{array}{llll} s_4, = -\operatorname{sn}(u_1 + u_2 + u_3), & = A_2A_3 s_1 + A_3A_1 s_2 + A_1A_2 s_3 & \div\, (A_2A_3 + A_3A_1 + A_1A_2) \\ c_4, = \quad \operatorname{cn}(\quad \text{,,} \quad) & = \;\text{,,}\; c_1 + \;\text{,,}\; c_2 + \;\text{,,}\; c_3 & \div\, (\qquad \text{,,} \qquad) \\ d_4, = \quad \operatorname{dn}(\quad \text{,,} \quad) & = \;\text{,,}\; d_1 + \;\text{,,}\; d_2 + \;\text{,,}\; d_3 & \div\, (\qquad \text{,,} \qquad) \end{array}$$

which are the required expressions. If $u_3 = 0$, and consequently $s_3, c_3, d_3 = 0, 1, 1$, the resulting expressions give the sn, cn, and dn of $u_1 + u_2$, but the expressions are in a very complicated form, not easily identifiable with the ordinary ones.

133. The determinant equation may be written

$$\begin{aligned}&(s_1 - s_2)(c_3 d_4 - c_4 d_3) + (s_3 - s_4)(c_1 d_2 - c_2 d_1)\\ &+ (c_1 - c_2)(d_3 s_4 - d_4 s_3) + (c_3 - c_4)(d_1 s_2 - d_2 s_1)\\ &+ (d_1 - d_2)(s_3 c_4 - s_4 c_3) + (d_3 - d_4)(s_1 c_2 - s_2 c_1) = 0,\end{aligned}$$

and in fact each of the three lines is separately $= 0$. This appears from the following three formulæ

$$\begin{aligned}\frac{\operatorname{sn}(u_1 + u_2)}{\operatorname{cn}(u_1 + u_2) - \operatorname{dn}(u_1 + u_2)} &= \frac{s_1 - s_2}{c_1 d_2 - c_2 d_1},\\ \frac{\operatorname{sn}(u_1 + u_2)}{\operatorname{cn}(u_1 + u_2) + 1} &= \frac{c_1 - c_2}{d_1 s_2 - d_2 s_1},\\ \frac{\operatorname{sn}(u_1 + u_2)}{\operatorname{dn}(u_1 + u_2) + 1} &= \frac{-\frac{1}{k^2}(d_1 - d_2)}{s_1 c_2 - s_2 c_1},\end{aligned}$$

which are themselves at once deducible from the formulæ,

$$\begin{aligned}&\operatorname{sn}(u_1 + u_2) = s_1^2 - s_2^2, = -(c_1^2 - c_2^2), = -\frac{1}{k^2}(d_1^2 - d_2^2), \div (s_1 c_2 d_2 - s_2 c_1 d_1),\\ &\operatorname{cn}(u_1 + u_2) = s_1 c_1 d_2 - s_2 c_2 d_1 \qquad \div \quad \text{Do.}\\ &\operatorname{dn}(u_1 + u_2) = s_1 d_1 c_2 - s_2 d_2 c_1 \qquad \div \quad \text{Do.}\end{aligned}$$

In fact the numerators of $\operatorname{cn}(u_1 + u_2) + \operatorname{dn}(u_1 + u_2)$, $\operatorname{cn}(u_1 + u_2) + 1$, $\operatorname{dn}(u_1 + u_2) + 1$ thus become $= (s_1 + s_2)(c_1 d_2 - c_2 d_1)$, $-(c_1 + c_2)(d_1 s_2 - d_2 s_1)$, $(d_1 + d_2)(s_1 c_2 - s_2 c_1)$ respectively: so that taking the numerator of $\operatorname{sn}(u_1 + u_2)$ under its three forms successively, we have by division the formulæ in question.

134. The above three equations, putting therein $u_1 + u_2 = -2u_3$, and reducing the functions of $2u_3$ become

$$\begin{aligned}\frac{1}{k'^2}\frac{c_3 d_3}{s_3} &= \frac{s_1 - s_2}{c_1 d_2 - c_2 d_1}, \text{ giving } -k^2 s_3^2 &&= \frac{(c_1 - c_2)(d_1 - d_2)}{(d_1 s_2 - d_2 s_1)(s_1 c_2 - s_2 c_1)}\\ -\frac{s_3 d_3}{c_3} &= \frac{c_1 - c_2}{d_1 s_2 - d_2 s_1}, \quad \text{“} \quad \frac{k^2}{k'^2} c_3^2 &&= \frac{(d_1 - d_2)(s_1 - s_2)}{(s_1 c_2 - s_2 c_1)(c_1 d_2 - c_2 d_1)}\\ \frac{k^2 s_3 c_3}{d_3} &= \frac{d_1 - d_2}{s_1 c_2 - s_2 c_1}, \quad \text{“} \quad -\frac{1}{k'^2} d_3^2 &&= \frac{(s_1 - s_2)(c_1 - c_2)}{(c_1 d_2 - c_2 d_1)(d_1 s_2 - d_2 s_1)},\end{aligned}$$

equations which must of course give each of them the same value for s_3^2: the equations belong to the relation $u_1 + u_2 + 2u_3 = 0$, viz. (s_3, c_3, d_3) are the coordinates of a point of contact of the tangent plane drawn through the two given points (s_1, c_1, d_1) and (s_2, c_2, d_2) of the curve.

135. Write

$$a, b, c, f, g, h = s_1 - s_2,\ c_1 - c_2,\ d_1 - d_2,\ c_1d_2 - c_2d_1,\ d_1s_2 - d_2s_1,\ s_1c_2 - s_2c_1,$$
$$a', b', c', f', g', h' = s_3 - s_4,\ c_3 - c_4,\ d_3 - d_4,\ c_3d_4 - c_4d_3,\ d_3s_4 - d_4s_3,\ s_3c_4 - s_4c_2,$$

so that (a, b, c, f, g, h) are the six coordinates of the line 12, and a', b', c', f', g', h' are the six coordinates of the line 34. The determinant equation is nothing else than the condition of the intersection of the two lines, viz. this is

$$af' + a'f + bg' + b'g + ch' + c'h = 0,$$

and by what precedes it appears that not only is this so but that we have separately $af' + a'f = 0$, $bg' + b'g = 0$, $ch' + c'h = 0$, viz. these three equations are satisfied by the coordinates of the lines 12 and 34 which join in pairs the intersections 1, 2 and 3, 4 of the quadri-quadric curve by a plane. But this is a geometrical property depending only on the four points being in a plane: and it is thus a result of Abel's theorem that when the arguments are such that

$$u_1 + u_2 + u_3 + u_4 = 0,$$

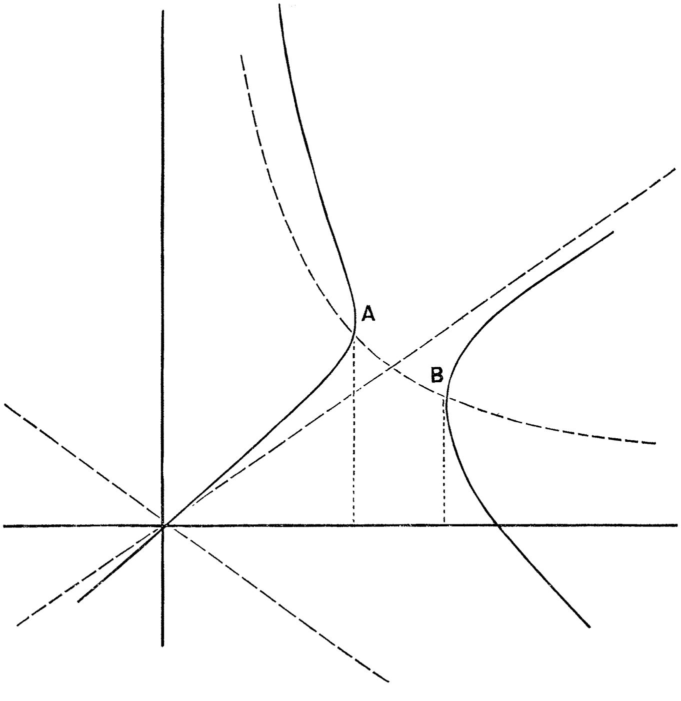
then not only the original equation, but each of the three equations, holds good.

The Cubic Curve $xy^2 - 2y + (1 + k^2)x - k^2x^3 = 0$. Art. Nos. 136 and 137.

136. Writing the equation in the forms

$$(xy - 1)^2 = (1 - x^2)(1 - k^2x^2), \text{ or say } xy - 1 = -\sqrt{1 - x^2 . 1 - k^2x^2}.$$
$$x(y^2 - k^2x^2) = 2y - (1 + k^2)x.$$

We see that the general form is as shown in the figure: the real portions of the



curve lie between the values $x = -\infty,\ -\frac{1}{k};\ -1,\ +1$; and $\frac{1}{k},\ \infty$. The curve may be made to depend on elliptic functions in two different ways: we may write

$$x = \operatorname{sn} u \qquad , = \frac{2i}{1+k}\,\frac{\operatorname{sn}_1 v}{\operatorname{cn}_1 v \operatorname{dn}_1 v},$$

$$y = \frac{1-\operatorname{cn} u \operatorname{dn} u}{\operatorname{sn} u},\ = \frac{i}{1+k}\,\frac{\operatorname{sn}_1 v}{\operatorname{cn}_1 v \operatorname{dn}_1 v}\{1+k^2-(1-k)^2\operatorname{sn}_1^2 v\},$$

where the functions sn, cn, dn belong to the modulus k, and the functions sn_1, cn_1, dn_1 to the modulus $\theta,\ = \frac{1-k}{1+k}$. The first mode is obvious; as to the second, observe that the formulæ give

$$y-kx = \frac{i}{1+k}\,\frac{\operatorname{sn}_1 v}{\operatorname{cn}_1 v \operatorname{dn}_1 v}(1-k)^2\operatorname{cn}_1^2 v,\ = \frac{i(1-k)^2}{1+k}\,\frac{\operatorname{sn}_1 v \operatorname{cn}_1 v}{\operatorname{dn}_1 v},$$

$$y+kx = \frac{i}{1+k}\,\frac{\operatorname{sn}_1 v}{\operatorname{cn}_1 v \operatorname{dn}_1 v}(1+k)^2\operatorname{dn}_1^2 v,\ = i(1+k)\,\frac{\operatorname{sn}_1 v \operatorname{dn}_1 v}{\operatorname{cn}_1 v},$$

whence $y^2-k^2x^2 = -(1-k)^2\operatorname{sn}_1^2 v$; and therefore

$$x(y^2-k^2x^2) = \frac{-2i(1-k)^2}{1+k}\,\frac{\operatorname{sn}_1^3 v}{\operatorname{cn}_1 v \operatorname{dn}_1 v},$$

which is also the value of $2y-(1+k^2)x$.

We find, moreover, $d\omega,\ = du,\ = \frac{2idv}{1+k}$; and thence, u, v each vanishing together, $u = \frac{2iv}{1+k}$. Writing for shortness s_1, c_1, d_1 to denote $\operatorname{sn}_1 v$, $\operatorname{cn}_1 v$, $\operatorname{dn}_1 v$ (that is s_1, c_1, d_1 are the elliptic functions of v to the modulus θ), we have

$$\operatorname{sn}\left(\frac{2iv}{1+k},\ k\right) = \frac{2i}{1+k}\,\frac{s_1}{c_1 d_1},$$

$$\operatorname{cn}\left(\frac{2iv}{1+k},\ k\right) = \frac{1}{(1+k)c_1 d_1}\{1+k+(1-k)s_1^2\},$$

$$\operatorname{dn}\left(\frac{2iv}{1+k},\ k\right) = \frac{1}{(1+k)c_1 d_1}\{1+k-(1-k)s_1^2\}.$$

137. To bring these into a known form, for k write $\frac{1-k}{1+k}$, then θ is changed into k and the formulæ become

$$\operatorname{sn}_1(1+k)iv = (1+k)i\frac{s}{cd},$$

$$\operatorname{cn}_1(1+k)iv = \frac{1}{cd}(1+ks^2),$$

$$\operatorname{dn}_1(1+k)iv = \frac{1}{cd}(1-ks^2),$$

where the sn_1, cn_1, dn_1 refer to the modulus θ, and s, c, d denote $\mathrm{sn}\, v$, $\mathrm{cn}\, v$, $\mathrm{dn}\, v$, modulus k.

But from the formulæ, p. 63, of my Elliptic Functions,

$$\mathrm{sn}\,(iu,\, k),\ \mathrm{cn}\,(iu,\, k),\ \mathrm{dn}\,(iu,\, k) = \frac{i\,\mathrm{sn}\,(u,\, k')}{\mathrm{cn}\,(u,\, k')},\ \frac{1}{\mathrm{cn}\,(u,\, k')},\ \frac{\mathrm{dn}\,(u,\, k')}{\mathrm{cn}\,(u,\, k')},$$

and herein for u writing $(1+k)v$, and for k writing θ, $= \dfrac{1-k}{1+k}$, whence k' becomes $\dfrac{2\sqrt{k}}{1+k}$, $=\gamma$, and $\gamma' = \dfrac{1-k}{1+k}$, we have

$$\mathrm{sn}_1(1+k)iv,\ \mathrm{cn}_1(1+k)iv,\ \mathrm{dn}_1(1+k)iv = \frac{i\,\mathrm{sn}(\overline{1+k}v,\gamma)}{\mathrm{cn}(\overline{1+k}v,\gamma)},\ \frac{1}{\mathrm{cn}(\overline{1+k}v,\gamma)},\ \frac{\mathrm{dn}(\overline{1+k}v,\gamma)}{\mathrm{cn}(\overline{1+k}v,\gamma)},$$

and the formulæ above obtained are

$$\frac{\mathrm{sn}(\overline{1+k}v,\gamma)}{\mathrm{cn}(\overline{1+k}v,\gamma)} = (1+k)\frac{s}{cd}, \qquad \text{giving } \mathrm{sn}(\overline{1+k}\,v,\ \gamma) = \frac{(1+k)s}{1+ks^2},$$

$$\frac{1}{\mathrm{cn}(\overline{1+k}v,\gamma)} = \frac{1}{cd}(1+ks^2), \qquad \mathrm{cn}(\overline{1+k}\,v,\ \gamma) = \frac{cd}{1+ks^2},$$

$$\frac{\mathrm{dn}(\overline{1+k}v,\gamma)}{\mathrm{cn}(\overline{1+k}v,\gamma)} = \frac{1}{cd}(1-ks^2), \qquad \mathrm{dn}(\overline{1+k}\,v,\ \gamma) = \frac{1-ks^2}{1+ks^2},$$

where as before s, c, d denote $\mathrm{sn}\,(v,\, k)$, $\mathrm{cn}\,(v,\, k)$, $\mathrm{dn}\,(v,\, k)$: these last are in fact the formulæ of the second line of the table, Elliptic Functions, p. 183:

Fixed Curve the Cubic $y^2 = x(1-x)(1-k^2x)$: *the Function* $\{01'\theta\}$.

Art. Nos. 138 to 142.

138. It was shown, No. 65, how for the affected theorem when the fixed curve is a cubic, the form of differential was $d\omega$ into

$$\{012\} + \Big[\int_2^1 d\omega\,\partial_3\{036\} - \{123\}\Big],$$

the last term being the properly determined constant K, attached to the variable term $\{012\}$, in order to obtain a standard form of integral. The object of the present article is to show what these formulæ become for the before-mentioned form of cubic curve $y^2 = x\,(1-x)(1-k^2x)$ which is most directly connected with elliptic functions, and to exhibit the connection of the formulæ with the ordinary formulæ for elliptic integrals of the second and third kinds respectively.

139. We have in general

$$\{012\}, = \frac{\widetilde{012}}{012}, = \left\{\begin{array}{l} (1+k^2)x_1x_2 - (x_1+x_2) + y_1y_1 \\ +\, x\,[-1 + (1+k^2)(x_1+x_2) \\ +\, y(y_1+y_2) \end{array}\right\} \div \left|\begin{array}{lll} x\,, & y\,, & 1 \\ x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \end{array}\right|.$$

Taking here 2, $=\theta$, for the point, coordinates $x_2,\ y_2 = 0,\ 0$; we have

$$\{01\theta\} = \frac{-x_1 - x + (1+k^2)xx_1 + yy_1}{xy_1 - x_1 y};$$

and if retaining 1 to denote the point coordinates $(x_1,\ y_1)$, belonging to the argument u_1, we write 1′ for the point belonging to the argument $u + iK'$, then the coordinates of 1′ are $\frac{1}{k^2 x_1},\ \frac{-y_1}{k^2 x_1^2}$, and the formula becomes

$$\{01'\theta\} = \frac{x_1 + k^2 x x_1^2 - (1+k^2)xx_1}{xy_1 + x_1 y};$$

the numerator hereof multiplied by x is $= y(xy_1 + x_1 y) - k^2x^2x_1(x - x_1)$, and we thence have

$$\{01'\theta\} = \frac{y}{x} - \frac{k^2 xx_1(x - x_1)}{xy_1 - x_1 y}$$

which substituting for $x,\ y,\ x_1,\ y_1$ their values in terms of $u,\ u_1$ is

$$= \frac{\operatorname{cn} u \operatorname{dn} u}{\operatorname{sn} u} - k^2 \operatorname{sn} u \operatorname{sn} u_1 \operatorname{sn}(u - u_1).$$

Operating on each side with $\frac{d}{du_1}$, $=\partial_1$, we obtain

$$\partial_1\{01'\theta\} = k^2\operatorname{sn}^2 u_1 - k^2 \operatorname{sn}^2(u - u_1),$$

the differentiation being in fact that which occurs in establishing the fundamental property of the elliptic integral of the second kind $Zu = u\left(1 - \frac{E}{K}\right) + k^2\int_0 \operatorname{sn}^2 u du$, viz. we have $Zu - Zu_1 - Z(u - u_1) = -k^2 \operatorname{sn} u \operatorname{sn} u_1 \operatorname{sn}(u - u_1)$, and thence $\partial_1[-k^2 \operatorname{sn} u \operatorname{sn} u_1 \operatorname{sn}(u - u_1)] = -Z'u_1 + Z'(u - u_1)$, $= k^2\operatorname{sn}^2 u_1 - k^2\operatorname{sn}^2(u - u_1)$. Observe that 1′, referring to $u_1 + ik'$, the subscript 1 might be written 1′.

The same result should of course be obtainable by the differentiation of the expression of $\{01'\theta\}$ in terms of x, y, x_1, y_1: we have

$$\partial_1 x_1 = 2y_1,\quad \partial_1 y_1 = 1 - 2(1+k^2)x_1 + 3k^2x_1^2,\ = \Omega_1$$

for a moment, and we thence obtain

$$\partial_1\{01'\theta\} = \frac{k^2}{(xy_1 + x_1 y)^2}[-2(xy_1 + x_1 y)(x - 2x_1)xy_1 + xx_1(x - x_1)(x\Omega_1 + 2yy_1)]$$

where the term in [ ] is found to be $= x(xy_1 + x_1 y)^2 - xx_1(x - x_1)^2$; whence the equation is

$$\partial_1\{01'\theta\} = k^2 x - \frac{k^2 xx_1(x - x_1)^2}{(xy_1 + x_1 y)^2},$$

giving the foregoing result.

140. To introduce into the formulæ 1 instead of 1′ we have only to write $u_1 - iK'$ instead of u_1; putting also for shortness $s,\ c,\ d,\ s_1,\ c_1,\ d_1$ for the functions of u and u_1 respectively, we thus obtain

$$\{01\theta\} = -\frac{cd}{s} + \frac{s}{s_1 \operatorname{sn}(u - u_1)},$$

where observe that interchanging u, u_1 we have

$$\{10\theta\} = -\frac{c_1 d_1}{s_1} - \frac{s_1}{s\,\mathrm{sn}\,(u-u_1)},$$

that is $\{10\theta\} = -\{01\theta\}$ as it should be: the formulæ may be written

$$\{01\theta\}, = -\{10\theta\}, = \frac{s^3 c_1 d_1 + s_1^3 cd}{sc_1 d_1 + s_1 cd}\,\frac{1}{ss_1\,\mathrm{sn}\,(u-u_1)}:$$

and

$$\partial_1\{01\theta\} = \frac{1}{s_1^2} - \frac{1}{\mathrm{sn}^2\,(u-u_1)},$$

whence

$$\partial_0\{01\theta\} = -\frac{1}{s^2} + \frac{1}{\mathrm{sn}^2\,(u-u_1)},$$

we have, moreover,

$$\{012\} = \{12\theta\} + \frac{s}{s_1\,\mathrm{sn}\,(u-u_1)} - \frac{s}{s_2\,\mathrm{sn}\,(u-u_1)},$$

and

$$\partial_0\{012\} = \frac{1}{\mathrm{sn}^2\,(u-u_1)} - \frac{1}{\mathrm{sn}^2\,(u-u_2)},$$

which last equation gives $(\partial_0 + \partial_1 + \partial_2)\{012\} = 0$ as it should do.

141. Supposing that the differential $d\Pi_{12}$ is defined by the equation

$$d\Pi_{12} = du\,\{012\} + du\Big[\int_2^1 du\,\partial_3\{036\} - \{123\}\Big],$$

we have

$$\int_5^4 d\Pi_{12} = \int_5^4 du\{012\} + \int_5^4 du\,.\Big[\int_2^1 du\,\partial_3\{036\} - \{123\},$$

and thence

$$\begin{aligned}\partial_1\partial_4\int_5^4 d\Pi_{12} &= \partial_1\{124\} + \partial_3\{136\} - \partial_1\{123\},\\ &= \partial_1\{134\} - \partial_3\{316\},\\ &= \Big[\frac{1}{\mathrm{sn}^2\,(u_1-u_3)} - \frac{1}{\mathrm{sn}^2\,(u_1-u_4)}\Big] - \Big[\frac{1}{\mathrm{sn}^2\,(u_3-u_1)} - \frac{1}{\mathrm{sn}^2\,(u_3-u_6)}\Big],\\ &= \frac{1}{\mathrm{sn}^2\,(u_3-u_6)} - \frac{1}{\mathrm{sn}^2\,(u_1-u_4)},\end{aligned}$$

or establishing between the constants u_3, u_6 the relation $\mathrm{sn}^2(u_3-u_6) = \dfrac{K}{K-E}$, this becomes

$$= 1 - \frac{E}{K} - k^2\,\mathrm{sn}^2\,(u_4 - u_1 + iK'),$$

which is

$$= \partial_2\partial_4\,.\log\frac{\theta(u_4-u_2+iK')\,\theta(u_5-u_1+iK')}{\theta(u_4-u_1+iK')\,\theta(u_5-u_2+iK')},$$

where Θ is Jacobi's theta-function, see Elliptic Functions, p. 144. The expression is in fact $= -\partial_1\partial_4\log\theta(u_4 - u_1 + iK')$, $= \phi(u_4 - u_1 + iK')$, if for a moment $\phi v = \partial_v^2\log\Theta v$. But we have

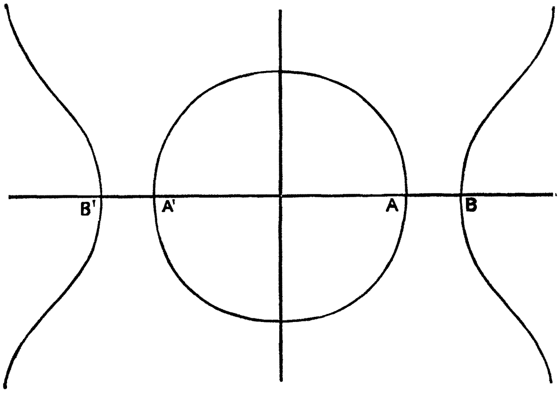
$$\log\Theta v = \log\sqrt{\frac{2k'K}{\pi}} + \tfrac{1}{2}v^2\Big(1 - \frac{E}{K}\Big) - k^2\int_0\int_0 \mathrm{sn}^2 v\,dv^2,$$

that is $\phi v = 1 - \frac{E}{K} - k^2 \operatorname{sn}^2 v$, and consequently we have $\phi(u_4 - u_1 + iK')$ is $= 1 - \frac{E}{K} - k^2 \operatorname{sn}^2(u_4 - u_1 + iK')$.

142. In connection with the same curve $y^2 = x(1-x)(1-k^2x)$, we may establish the identity
$$\frac{d}{du_1}\frac{y_1}{x_1 - x} + \frac{d}{du}\frac{y}{x_1 - x} = k^2(x_1 - x),$$
where as before $x, y = s^2, scd$, and $x_1, y_1 = s_1^2, s_1c_1d_1$. We have
$$(x_1 - x)\frac{dy_1}{du_1} - y_1\frac{dx_1}{du_1} = (s_1^2 - s^2)\{1 - 2(1+k^2)s_1^2 + 3k^2s_1^4\} - 2s_1^2(1-s_1^2)(1-k^2s_1^2)$$
$$= -s^2 - s_1^2 + 2(1+k^2)s^2s_1^2 - 3k^2s^2s_1^4 + k^2s_1^6;$$
and similarly
$$(x - x_1)\frac{dy}{du} - y\frac{dx}{du} = -s^2 - s_1^2 + 2(1+k^2)s^2s_1^2 - 3k^2s^4s_1^2 + k^2s^6.$$
The difference of the two functions on the right-hand side is $= k^2(s_1^2 - s^2)^3$; which is $= k^2(x_1 - x)^3$, and this divided by $(x_1 - x)^2$ is $= k^2(x_1 - x)$; the identity is thus verified.

Fixed Curve the Quartic $y^2 = (1-x^2)(1-k^2x^2)$. Art. Nos. 143 to 145.

143. This is a curve having a tacnode at infinity on the line $x = 0$, as may be seen by writing the equation in the homogeneous form $y^2z^2 = (z^2 - x^2)(z^2 - k^2x^2)$; we have as it were two branches having the line infinity for a common tangent at the point in question. The equation is satisfied by $x = \operatorname{sn} u$, $y = \operatorname{cn} u \operatorname{dn} u$, values which are unaltered by the change of u into $u + 4mK + 2m'iK'$, m and m' any positive or negative integers, and in regard to this curve the sign $\equiv$ is to be understood accordingly. I consider with reference to this curve only the affected theorem, in the particular form in which it most readily connects itself with the ordinary theory of the integral of the third kind.



144. I consider the differential $\frac{(x, y, z)_{12}^2\, d\omega}{012}$ in the particular case where the line 012 is a line parallel to the axis of y: taking its equation to be $x - x_1 = 0$, and putting for shortness $X = \sqrt{1 - x^2 . 1 - k^2x^2}$, $X_1 = \sqrt{1 - x_1^2 . 1 - k^2x_1^2}$, the para-

metric points are taken to be $(x_1, \sqrt{X_1})$, $(x_1, -\sqrt{X_1})$, and the residues are the intersections with the two branches at the tacnode. The conic $(x, y, z)^2_{12} = 0$ is to satisfy the conditions of passing through the two nodes of the tacnode, and through the two residues, that is again through the tacnode twice—in all four conditions, and we have thus the form $z(x - \theta z) = 0$, containing the arbitrary constant θ: the major function itself is then easily determined, and putting again $z = 1$ we arrive at the form $\dfrac{\sqrt{X_1}}{x_1 - \theta}\dfrac{x-\theta}{x-x_1}\dfrac{dx}{\sqrt{X}}$.

If the limits are taken to be two points on a line parallel to the axis of x, or what is the same thing, if the limits in regard to x are $x, -x$, we have the integral

$$\int_{-x}^{x}\frac{\sqrt{X_1}}{x_1-\theta}\,\frac{x-\theta}{x-x_1}\,\frac{dx}{\sqrt{X}}, = \int_0^x \frac{\sqrt{X_1}}{x_1-\theta}\left(\frac{x-\theta}{x-x_1}+\frac{x+\theta}{x+x_1}\right)\frac{dx}{\sqrt{X}},$$

$$= 2\int_0^x \frac{\sqrt{X_1}}{x_1-\theta}\cdot\frac{x^2-x_1\theta}{x^2-x_1^2}\,\frac{dx}{\sqrt{X}}.$$

We have $\dfrac{1}{x_1}\left(\dfrac{x^2}{x_1^2-x^2}-\dfrac{\theta}{x_1-\theta}\right)=\dfrac{x^2-x_1\theta}{x_1^2-x^2.x_1-\theta}$, and the integral thus becomes

$$= -2\int_0^x \frac{\sqrt{X_1}x^2dx}{x_1(x_1^2-x^2)\sqrt{X}} + 2\frac{\theta\sqrt{X_1}}{x_1(x_1-\theta)}\int_0^x\frac{dx}{\sqrt{X}}.$$

Taking here $x = \text{sn}\,u$, $x_1 = \text{sn}(a + iK')$, $= \dfrac{1}{k\,\text{sn}\,a}$, we have $dx = \text{cn}\,u\ \text{dn}\,u\,du = \sqrt{X}\,du$, and the result is

$$\int_{-x}^{x}\frac{\sqrt{X_1}}{x_1-\theta}\,\frac{x-\theta}{x-x_1}\,\frac{dx}{\sqrt{X}} = -2\int_0 \frac{k^2\,\text{sn}\,a\,\text{cn}\,a\,\text{dn}\,a\,\text{sn}^2u\,du}{1-k^2\,\text{sn}^2a\,\text{sn}^2u} + \frac{2k\,\text{sn}\,a\,\text{cn}\,a}{1-k\theta\,\text{sn}\,a}u,$$

where on the right-hand side the first term is $= -2\Pi(u, a)$, if $\Pi(u, a)$ be Jacobi's form of the integral of the third kind, Elliptic Functions, p. 143.

145. It is to be observed that the proper normal form is not $\Pi(u, a)$, but $\Pi(u, a) - uZa$; say this is $\overline{\Pi}(u, a)$, viz. we then have

$$\overline{\Pi}(u, a) = \Pi(u, a) - u\left[a\left(1-\frac{E}{K}\right) - k^2\int \text{sn}^2a\,da\right],$$

and thence

$$\partial_u\overline{\Pi}(u, a) = \frac{k^2\,\text{sn}\,a\,\text{cn}\,a\,\text{dn}\,a\,\text{sn}^2u}{1-k^2\,\text{sn}^2a\,\text{sn}^2u} - a\left(1-\frac{E}{K}\right) + k^2\int\text{sn}^2a\,da,$$

$$\partial_a\partial_u\overline{\Pi}(u, a) = k^2\,\text{sn}^2u\,\partial_a\frac{\text{sn}\,a\,\text{cn}\,a\,\text{dn}\,a}{1-k^2\,\text{sn}^2a\,\text{sn}^2u} - \left(1-\frac{E}{K}\right) + k^2\,\text{sn}^2a;$$

or if for shortness we write $\text{sn}\,u$, $\text{sn}\,a = s, \sigma$, this is

$$\partial_a\partial_u\overline{\Pi}(u, a) = \frac{k^2s^2[1-2(1+k^2)\sigma^2+3k^2\sigma^4]+k^4s^4\sigma^2}{(1-k^2s^2\sigma^2)^2} - \left(1-\frac{E}{K}\right) + k^2\sigma^2,$$

which is

$$= \frac{k^2\{(s^2+\sigma^2)(1+k^2s^2\sigma^2)-2(1+k^2)s^2\sigma^2\}}{(1-k^2s^2\sigma^2)^2} - \left(1-\frac{E}{K}\right);$$

or this being symmetrical in regard to s, σ, we have

$$\partial_a\partial_u\overline{\Pi}(u, a) = \partial_a\partial_u\overline{\Pi}(a, u),$$

and thence by integration, and a proper determination of the constants,

$$\overline{\Pi}(u, a) = \overline{\Pi}(a, u).$$

CHAPTER VI. THE NODAL QUARTIC.

Nodal Quartic; the General and Fleflecnodal Forms. Art. Nos. 146 to 148.

146. For a cubic, or other curve of deficiency 1, we are concerned with single points on the curve, and corresponding thereto with functions of a single argument (elliptic functions): but for a curve of deficiency 2, we have to consider pairs of points on the curve, and functions of two arguments: there is thus a marked change in the form of the results.

The most simple curve of deficiency 2 is the nodal quartic, $n = 4$, $p = 2$. Using homogeneous coordinates the general form is $Az^2 + 2Bz + C = 0$, where

$$\begin{aligned} A &= (i, j, k \between x, y)^2, \\ B &= (l, m, n, o \between x, y)^3, \\ C &= (p, q, r, s, t \between x, y)^4, \end{aligned}$$

and where we write also

$$B^2 - AC = (l^2 - ip)(x - ay)(x - by)(x - cy)(x - dy)(x - ey)(x - fy).$$

Clearly the equation of the two tangents at the node is $A = 0$; and the equations of the six tangents from the node are $x - ay = 0, \ldots x - fy = 0$: at the points of contact we have $Az + B = 0$, viz. this is the equation of a nodal cubic, the node and the two tangents there being the same with the node and two tangents of the quartic. Hence the node counts as 6 intersections, and there are besides 6 intersections which are the points of contact of the 6 tangents respectively: say these are the points a, b, c, d, e, f: the coordinates of the point a are given by the equations $x : y : z = a : 1 : -\frac{B_a}{A_a}\left(= -\frac{C_a}{B_a}\right)$, where A_a, B_a, C_a are what A, B, C become on writing therein a, 1 for x, y: and similarly for the other points.

147. An important special case is when the equation is $B = 0$; say we have here

$$\begin{aligned} A &= i(x - ey)(x - fy), \\ B &= 0, \\ C &= p(x - ay)(x - by)(x - cy)(x - dy), \end{aligned}$$

or omitting the factors i and p,

$$(x - ey)(x - fy)z^2 = (x - ay)(x - by)(x - cy)(x - dy),$$

the origin is here a fleflecnode; the tangents $x - ey = 0$, $x - fy = 0$ count as two of the six tangents from the node, and there remain the four tangents

$$x - cy = 0,\ x - dy = 0,\ x - ay = 0,\ x - by = 0;$$

the four points of contact are the intersections of the curve with the line $z = 0$.

148. The general nodal form depends on 11 constants, but by writing $\alpha x + \beta y$, $\gamma x + \delta y$, εz in place of x, y, z, we introduce 5 apoclastic constants, and so reduce the number to $11 + 1 - 5$, $= 7$: similarly the fleflecnodal form depends on 7 constants, but we reduce the number in like manner to $7 + 1 - 5$, $= 3$: the final form might here be taken to be

$$z^2 xy = (x - y)(x - by)(x - cy)(x - dy);$$

but it is more convenient to retain the original form

$$z^2(x - ey)(x - fy) = (x - ay)(x - by)(x - cy)(x - dy),$$

bearing in mind that this is reducible to the form just referred to, and thus depends virtually upon only 3 constants.

It is a general property that a curve of deficiency p greater than 1 can be by a rational transformation reduced to a curve of that deficiency depending upon $3p - 3$ parameters: in particular if $p = 2$, then the form depending upon 3 parameters may be taken to be the fleflecnodal quartic as above: and I proceed to show how the general nodal quartic can in fact be reduced to this fleflecnodal form.

Reduction to the Fleflecnodal Form. Art. Nos. 149 to 152.

149. Consider the general nodal quartic $Az^2 + 2Bz + C = 0$: take $\zeta = 0$ for the equation of the line joining the points of contacts of the tangents $x - ey = 0$, $x - fy = 0$; and then writing $x = \xi$, $y = \eta$, let the curve be transformed in the first instance from x, y, z to the new coordinates ξ, η, ζ.

Writing A_e for the value $(i, j, k \between e, 1)^2$ which A assumes on putting therein $(e, 1)$ for (x, y) respectively, and similarly A_f, B_e, B_f for the other like values, we may take

$$A_2 A_f (e - f)\zeta = \begin{vmatrix} x\ , & y\ , & z \\ eA_e, & A_e, & -B_e \\ fA_f, & A_f, & -B_f \end{vmatrix},$$

$$= -x(A_e B_f - A_f B_e) + y(eA_e B_f - fB_e A_f) + z(e - f) A_e A_f,$$

say this equation is $\zeta = -\lambda x - \mu y + z$, the values of λ, μ being

$$\lambda = \frac{A_e B_f - A_f B_e}{(e - f) A_e A_f}, \qquad \mu = \frac{-eA_e B_f + fB_e A_f}{(e - f) A_e A_f},$$

and therefore

$$\lambda e + \mu = -\frac{B_e}{A_e}, \qquad \lambda f + \mu = -\frac{B_f}{A_f}.$$

150. From the values $\zeta, \xi, \eta = -\lambda x - \mu y + z, x, y$, we obtain $z, x, y = \zeta + \lambda\xi + \mu\eta, \xi, \eta$; and the transformed equation is

$$A'(\zeta + \lambda\xi + \mu\eta)^2 + 2B'(\zeta + \lambda\xi + \mu\eta) + C' = 0,$$

where

$$A' = (i, j, k \between \xi, \eta)^2,$$
$$B' = (l, m, n, o \between \xi, \eta)^3,$$
$$C' = (p, q, r, s, t \between \xi, \eta)^4,$$

say this equation is $\mathfrak{A}\zeta^2 + 2\mathfrak{B}\zeta + \mathfrak{C} = 0$, where

$$\mathfrak{A} = A',$$
$$\mathfrak{B} = A'(\lambda\xi + \mu\eta) + B',$$
$$\mathfrak{C} = A'(\lambda\xi + \mu\eta)^2 + 2B'(\lambda\xi + \mu\eta) + C',$$

and thence

$$\mathfrak{B}^2 - \mathfrak{A}\mathfrak{C} = B'^2 - A'C', = (l^2 - ip)(\xi - a\eta)(\xi - b\eta)(\xi - c\eta)(\xi - d\eta)(\xi - e\eta)(\xi - f\eta):$$

We have here $\mathfrak{B}, = A'(\lambda\xi + \mu\eta) + B'$, a cubic function $(\xi, \eta)^3$ containing the factors $\xi - e\eta$ and $\xi - f\eta$: in fact writing $\xi, \eta = e, 1$ it becomes $A_e(\lambda e + \mu) + B_e$ which is $= 0$; and similarly writing $\xi, \eta = f, 1$ it becomes $A_f(\lambda f + \mu) + B_f$, which is $= 0$. Calling the other factor $L\xi + M\eta$, we have thus

$$\mathfrak{B} = (\xi - e\eta)(\xi - f\eta)(L\xi + M\eta),$$

and thence

$$\begin{aligned}\mathfrak{A}\mathfrak{C} &= (\xi - e\eta)^2(\xi - f\eta)^2(L\xi + M\eta)^2 \\ &\qquad - (l^2 - ip)(\xi - a\eta)(\xi - b\eta)(\xi - c\eta)(\xi - d\eta)(\xi - e\eta)(\xi - f\eta), \\ &= (\xi - e\eta)(\xi - f\eta)[(\xi - e\eta)(\xi - f\eta)(L\xi + M\eta)^2 \\ &\qquad - (l^2 - ip)(\xi - c\eta)(\xi - d\eta)(\xi - e\eta)(\xi - f\eta)].\end{aligned}$$

Hence $\mathfrak{C}$ contains the factor $(\xi - e\eta)(\xi - f\eta)$, say we have

$$\mathfrak{C} = \theta(\xi - e\eta)(\xi - f\eta)(\xi - \varepsilon\eta)(\xi - \phi\eta).$$

151. In the equation $\mathfrak{A}\zeta^2 + 2\mathfrak{B}\zeta + \mathfrak{C} = 0$ of the quartic curve, writing $\zeta = 0$, we find $\mathfrak{C} = 0$, that is $(\xi - e\eta)(\xi - f\eta)(\xi - \varepsilon\eta)(\xi - \phi\eta) = 0$: but $\zeta = 0$ is the equation of the line joining the points of contact of the tangents $\xi - e\eta = 0$, $\xi - f\eta = 0$; hence $\xi - \varepsilon\eta = 0$, $\xi - \phi\eta = 0$ are the lines drawn from the node to the two points ε, ϕ which are the residues of these two points of contact. We now have

$$\begin{aligned}\theta\mathfrak{A}(\xi - \varepsilon\eta)(\xi - \phi\eta) &= (\xi - e\eta)(\xi - f\eta)(L\xi + M\eta)^2 \\ &\qquad - (l^2 - ip)(\xi - a\eta)(\xi - b\eta)(\xi - c\eta)(\xi - d\eta),\end{aligned}$$

and thence

$$0 = (\varepsilon - e)(\varepsilon - f)(L\varepsilon + M)^2 - (l^2 - ip)(\varepsilon - a)(\varepsilon - b)(\varepsilon - c)(\varepsilon - d),$$
$$0 = (\phi - e)(\phi - f)(L\phi + M)^2 - (l^2 - ip)(\phi - a)(\phi - b)(\phi - c)(\phi - d),$$

which equations determine L and M; and then with these values of L, M, and for $\mathfrak{A}$ substituting its value $(i, j, k \between \xi, \eta)^2$ the equation must become an identity.

We have in what precedes, by the transformation $z = \zeta + \lambda\xi + \mu\eta$, $x = \xi$, $y = \eta$, passed from the form $Az^2 + 2Bz + C = 0$, to the form

$$\mathfrak{A}\zeta^2 + 2\mathfrak{B}\zeta + \mathfrak{C} = 0,$$

where

$$\begin{aligned} \mathfrak{A} &= (i,\ j,\ k)(\xi,\ \eta)^2, \\ \mathfrak{B} &= (\xi - e\eta)(\xi - f\eta)(L\xi + M\eta), \\ \mathfrak{C} &= \theta(\xi - e\eta)(\xi - f\eta)(\xi - \varepsilon\eta)(\xi - \phi\eta), \end{aligned}$$

viz. $\mathfrak{B}$ and $\mathfrak{C}$ have here the common factor $(\xi - e\eta)(\xi - f\eta)$.

152. Assume now

$$\xi,\ \eta,\ \zeta = X,\ Y,\ \theta\frac{(X - \varepsilon Y)(X - \varphi Y)}{Z - LX - MY},$$

and therefore conversely

$$X,\ Y,\ Z = \xi,\ \eta,\ L\xi + M\eta + \theta\frac{(\xi - \varepsilon\eta)(\xi - \varphi\eta)}{\zeta},$$

then we have in the new coordinates $(X,\ Y,\ Z)$ the equation

$$\begin{aligned} &(i,\ j,\ k)(X,\ Y)^2\theta^2\left\{\frac{(X - \varepsilon Y)(X - \varphi Y)}{Z - LX - MY}\right\}^2 \\ &+ 2(X - eY)(X - fY)(LX + MY)\,\theta\frac{(X - \varepsilon Y)(X - \varphi Y)}{Z - LX - MY}, \\ &+ \theta(X - eY)(X - fY)(X - \varepsilon Y)(X - \phi Y) = 0, \end{aligned}$$

that is

$$\begin{aligned} &(i,\ j,\ k)(X,\ Y)^2\theta(X - \varepsilon Y)(X - \phi Y) \\ &+ 2(X - eY)(X - fY)(LX + MY)(Z - LX - MY) \\ &+ (X - eY)(X - fY)(Z - LX - MY)^2 = 0, \end{aligned}$$

where the second and third lines together are

$$= (X - eY)(X - fY)\{Z^2 - (LX + MY)^2\},$$

and the equation thus is

$$\begin{aligned} (X - eY)(X - fY)Z^2 + \{\theta(i,\ j,\ k)(X,\ Y)^2(X - \varepsilon Y)(X - \phi Y)& \\ - (X - eY)(X - fY)(LX + MY)^2\} = 0.& \end{aligned}$$

But the term in { } is identically

$$= -(l^2 - ip)(X - aY)(X - bY)(X - cY)(X - dY),$$

and the equation thus becomes

$$(X - eY)(X - fY)Z^2 - (l^2 - ip)(X - aY)(X - bY)(X - cY)(X - dY) = 0;$$

viz. the original equation $Az^2 + 2Bz + C = 0$ of the general nodal quartic is, by the equations

$$x,\ y,\ z = X,\ Y,\ \lambda X + \mu Y + \theta\frac{(X - \varepsilon Y)(X - \varphi Y)}{Z - LX - MY};$$

or conversely

$$X,\ Y,\ Z = x,\ y,\ Lx + My + \frac{\theta(x - \varepsilon y)(x - \varphi y)}{z - \lambda\xi - \mu\eta},$$

transformed into the fleflecnodal form as above.

It originally appeared to me that the fleflecnodal form was more easily dealt with than the general form; and I effected the transformation for this reason: there is, however, the disadvantage that the six points a, b, c, d, e, f enter into the equation unsymmetrically; and I afterwards found that the general form could be dealt with nearly as easily, and in what follows I use therefore the general form. The transformation is given as interesting for its own sake, and as an illustration of the theorem in regard to the number of constants in a curve of deficiency p.

Application of Abel's Theorem. Art. Nos. 153 to 157.

153. Taking the fixed curve to be $f, = \frac{1}{2}(Az^2 + 2Bz + C), = 0$, we have $\frac{df}{dz} = Az + B = \sqrt{(x, y)^6}$, if for shortness we write

$$(x, y)^6 = B^2 - AC, = (l^2 - ip)(x - ay)(x - by)(x - cy)(x - dy)(x - ey)(x - fy),$$

and we thence have

$$d\omega = \frac{xdy - ydx}{\sqrt{(x, y)^6}}.$$

The minor curve $(x, y, z)^{n-3} = 0$, is an arbitrary line passing through the node, that is the point $x = 0$, $y = 0$; and the pure theorem thus gives the two relations $\Sigma x d\omega = 0$, $\Sigma y d\omega = 0$; where the summation extends to the intersections of the fixed curve $Az^2 + 2Bz + C = 0$, with the variable curve ϕ.

The variable curve is taken to be a cubic $Az + B = (\alpha, \beta, \gamma, \delta \between x, y)^3$, or say $Az + B = \Omega$, where Ω is a given cubic function $(x, y)^3$: viz. this is a nodal cubic, the node and the two tangents there being the same with the node and the two tangents of the quartic: hence it meets the quartic in the node counting 6 times, and in 6 other points, say these are the points 1, 2, 3, 4, 5, 6: hence the differential relations are

$$x_1 d\omega_1 + x_2 d\omega_2 + x_3 d\omega_3 + x_4 d\omega_4 + x_5 d\omega_5 + x_6 d\omega_6 = 0,$$
$$y_1 d\omega_1 + y_2 d\omega_2 + y_3 d\omega_3 + y_4 d\omega_4 + y_5 d\omega_5 + y_6 d\omega_6 = 0.$$

154. Observe that the intersections of the cubic with the fixed curve are given by the equation $\Omega^2 = B^2 - AC$, or say $\Omega^2 = (x, y)^6$; an equation which determines the ratio $x : y$ for the six points respectively, and the ratio $z : x$ is then determined rationally by the original equation $Az + B = \Omega$. Instead of regarding Ω as a given function, we may, if we please, take 1, 2, 3, 4 given points on the quartic: we then have four equations for the determination of the coefficients $(\alpha, \beta, \gamma, \delta)$ of the function Ω; viz. these equations may be taken to be

$$\begin{aligned}(\alpha,\ \beta,\ \gamma,\ \delta)(x_1,\ y_1)^3 &= \sqrt{(x_1,\ y_1)^6},\\ (\quad ,, \quad)(x_2,\ y_2)^3 &= \sqrt{(x_2,\ y_2)^6},\\ (\quad ,, \quad)(x_3,\ y_3)^3 &= \sqrt{(x_3,\ y_3)^6},\\ (\quad ,, \quad)(x_4,\ y_4)^4 &= \sqrt{(x_4,\ y_4)^6},\end{aligned}$$

Ω is hereby completely determined, and this being so the remaining intersections 5 and 6 are also completely determined: there are thus between the six points 2 integral relations, which agrees with the number, $= 2$, of the differential relations obtained above.

155. If we now assume

$$du = x_1 d\omega_1 + x_2 d\omega_2,\ du' = x_3 d\omega_3 + x_4 d\omega_4,\ du'' = x_5 d\omega_5 + x_6 d\omega_6,$$
$$dv = y_1 d\omega_1 + y_2 d\omega_2,\ dv' = y_3 d\omega_3 + y_4 d\omega_4,\ dv'' = y_5 d\omega_5 + y_6 d\omega_6,$$

or say

$$u,\ u',\ u'' = \int_{\alpha\beta}^{12},\ \int_{\alpha\beta}^{34},\ \int_{\alpha\beta}^{56} \frac{x(xdy - ydx)}{\sqrt{(x,\ y)^6}},$$
$$v,\ v',\ v'' = \int_{\alpha\beta}^{12},\ \int_{\alpha\beta}^{34},\ \int_{\alpha\beta}^{56} \frac{y(xdy - ydx)}{\sqrt{(x,\ y)^6}};$$

that is

$$u = \left(\int_{\alpha}^{1} + \int_{\beta}^{2}\right) \frac{x(xdy - ydx)}{\sqrt{(x,\ y)^6}},\ v = \left(\int_{\alpha}^{1} + \int_{\beta}^{2}\right) \frac{y(xdy - ydx)}{\sqrt{(x,\ y)^6}},$$

where α, β are points assumed at pleasure on the quartic: and similarly for u', v', u'', v'': then u, v are hereby determined as functions of the points 1, 2: and we may conversely regard the points 1, 2 as determined in terms of the two arguments u, v. We might, selecting any two symmetrical functions of the degree zero, for instance, $\frac{x_1}{y_1} + \frac{x_2}{y_2}$, $\frac{x_1 x_2}{y_1 y_2}$, represent them as functions $\phi(u,\ v)$, $\psi(u,\ v)$; and then $\frac{x_1}{y_1}$ and $\frac{x_2}{y_2}$ will be functions of $\phi(u,\ v)$, $\psi(u,\ v)$, but instead of this selection it is proper to consider the ratios of six functions depending on the points a, b, c, d, e, f respectively: viz. we assume

$$\begin{array}{ccccc}\sqrt{(x_1 - ay_1)(x_2 - ay_2)} : & \sqrt{(x_1 - by_1)(x_2 - by_2)} & \ldots : & \sqrt{x_1 - fy_1)(x_2 - fy_2)} \\ = \quad A(u,\ v) \quad : & B(u,\ v) & \ldots : & F(u,\ v),\end{array}$$

and of course 3, 4 will be in like manner determined by means of the corresponding functions of u', v', and 5, 6 by means of the corresponding functions of u'', v''. The squared functions A^2, B^2, C^2, D^2, E^2, F^2 are proportional to given linear functions of $x_1 x_2$, $x_1 y_2 + x_2 y_1$, $y_1 y_2$, and are thus connected by three independent linear relations.

156. The differential relations then become

$$du + du' + du'' = 0,\ dv + dv' + dv'' = 0,$$

and we have consequently

$$u + u' + u'' = I,\quad v + v' + v'' = J,$$

where I, J are constants which are determinable as definite integrals by the consideration that when the cubic is taken to be $Az+B=0$, the six points 1, 2, 3, 4, 5, 6 coincide with the points of contact a, b, c, d, e, f. I do not at present see my way to a proper development of this point of the theory: but in explanation of the nature of the result, I assume for the moment that by a proper determination of the inferior limits α, β, or otherwise, we may take $I=0$, $J=0$. We then have $u''=-u-u'$, $v''=-v-v'$; and the integral equations which determine the points 5, 6 in terms of the points 1, 2 and the points 3, 4, then in effect determine the functions A, B, etc. of $-u-u'$, $-v-v'$, or say those of $u+u'$, $v+v'$ in terms of the like functions of (u, v) and of (u', v'): viz. these equations give the addition theory of the functions $A(u, v)$, etc.

157. We may, in the first instance, disregarding altogether the consideration of the arguments u, v, etc., attend only to the algebraic functions such as $\sqrt{(x_1-ay_1)(x_2-ay_2)}$, etc. of the coordinates of the pairs of points 1, 2; 3, 4, and 5, 6; and we can in regard to these develope a proper theory. This depends only on the equation $\Omega=\sqrt{(x, y)^6}$; it will be convenient to assume herein $y=1$, and slightly modifying the form, to write it

$$(\alpha, \beta, \gamma, \delta)(x, 1)^3=\sqrt{a-x.b-x.c-x.d-x.e-x.f-x};$$

and accordingly to consider the functions $\sqrt{a-x_1.a-x_2}$, etc. These are called the single-letter functions A, etc. but there are certain double-letter functions AB, etc. which have also to be considered; and I will, in the first instance, show how these present themselves in connection with the cubic curve.

Origin of the Double-Letter Functions. Art. Nos. 158, 159.

158. The cubic curve $Az+B=\Omega$ may be taken to be a curve through two of the points of contact, say the points a, b; these will then be two out of the six points, and taking the remaining four points to be the pairs 1, 2 and 3, 4, we have single-letter functions of 3, 4 presenting themselves as double-letter functions of 1, 2. In fact the equation of the curve is

$$Az+B=\lambda(x-ay)(x-by)(x-ky);$$

for the intersections with the quartic we have $\lambda^2(x-ay)^2(x-by)^2(x-ky)^2=\Omega^2$, or throwing out the factor $(x-ay)(x-by)$ and changing the constant λ, this is $(x-ay)(x-by)(x-ky)^2-\lambda(x-cy)(x-dy)(x-ey)(x-fy)=0$; and the quartic function must be a multiple of $(xy_1-x_1y)(xy_2-x_2y)(xy_3-x_3y)(xy_4-x_4y)$.

Putting each of the y's equal 1, we have the identity

$$(a-x)(b-x)(k-x)^2-\lambda\,(c-x)(d-x)(e-x)(f-x)$$
$$=\mu\,(x_1-x)(x_2-x)(x_3-x)(x_4-x)\,;$$

and hence, introducing a notation which will be convenient, $a-x=\mathrm{a}$, $a-x_1=\mathrm{a}_1$, and so in other cases, we have by giving different values to x the equations

$$\begin{array}{ll}
\mathrm{a}_1\mathrm{b}_1\mathrm{k}_1^2=\lambda\mathrm{c}_1\mathrm{d}_1\mathrm{e}_1\mathrm{f}_1, & (a-c)(b-c)(k-c)^2=\mu\mathrm{c}_1\mathrm{c}_2\mathrm{c}_3\mathrm{c}_4,\\
\mathrm{a}_2\mathrm{b}_2\mathrm{k}_2^2=\lambda\mathrm{c}_2\mathrm{d}_2\mathrm{e}_2\mathrm{f}_2, & (a-d)(b-d)(k-d)^2=\mu\mathrm{d}_1\mathrm{d}_2\mathrm{d}_3\mathrm{d}_4,\\
\mathrm{a}_3\mathrm{b}_3\mathrm{k}_3^2=\lambda\mathrm{c}_3\mathrm{d}_3\mathrm{e}_3\mathrm{f}_3, & (a-e)(b-e)(k-e)^2=\mu\mathrm{e}_1\mathrm{e}_2\mathrm{e}_3\mathrm{e}_4,\\
\mathrm{a}_4\mathrm{b}_4\mathrm{k}_4^2=\lambda\mathrm{c}_4\mathrm{d}_4\mathrm{e}_4\mathrm{f}_4, & (a-f)(b-f)(k-f)^2=\mu\mathrm{f}_1\mathrm{f}_2\mathrm{f}_3\mathrm{f}_4,\\
 & -\lambda(c-a)(d-a)(e-a)(f-a)=\mu\mathrm{a}_1\mathrm{a}_2\mathrm{a}_3\mathrm{a}_4,\\
 & -\lambda(c-b)(d-b)(e-b)(f-b)=\mu\mathrm{b}_1\mathrm{b}_2\mathrm{b}_3\mathrm{b}_4.
\end{array}$$

We have thus
$$\frac{(a-c)(b-c)}{(a-d)(b-d)}\left(\frac{c-k}{d-k}\right)^2=\frac{\mathrm{c}_1\mathrm{c}_2\mathrm{c}_3\mathrm{c}_4}{\mathrm{d}_1\mathrm{d}_2\mathrm{d}_3\mathrm{d}_4},$$

and
$$\frac{\mathrm{k}_1^2}{\mathrm{k}_2^2}=\frac{\mathrm{a}_2\mathrm{b}_2\mathrm{c}_1\mathrm{d}_1\mathrm{e}_1\mathrm{f}_1}{\mathrm{a}_1\mathrm{b}_1\mathrm{c}_2\mathrm{d}_2\mathrm{e}_2\mathrm{f}_2},$$

which last equation, writing for a moment $\gamma,\ \delta=\sqrt{\mathrm{a}_2\mathrm{b}_2\mathrm{c}_1\mathrm{d}_1\mathrm{e}_1\mathrm{f}_1},\ \sqrt{\mathrm{a}_1\mathrm{b}_1\mathrm{c}_2\mathrm{d}_2\mathrm{e}_2\mathrm{f}_2}$, gives $\dfrac{k-x_1}{k-x_2}=\dfrac{\gamma}{\delta}$, whence $k(\gamma-\delta)=x_2\gamma-x_1\delta$, and thence

$$\frac{c-k}{d-k}=\frac{\gamma\mathrm{c}_2-\delta\mathrm{c}_1}{\gamma\mathrm{d}_2-\delta\mathrm{d}_1}=\frac{\sqrt{\mathrm{c}_1\mathrm{c}_2}\,\{\sqrt{\mathrm{a}_2\mathrm{b}_2\mathrm{c}_2\mathrm{d}_1\mathrm{e}_1\mathrm{f}_1}-\sqrt{\mathrm{a}_1\mathrm{b}_1\mathrm{c}_1\mathrm{d}_2\mathrm{e}_2\mathrm{f}_2}\}}{\sqrt{\mathrm{d}_1\mathrm{d}_2}\,\{\sqrt{\mathrm{a}_2\mathrm{b}_2\mathrm{d}_2\mathrm{c}_1\mathrm{e}_1\mathrm{f}_1}-\sqrt{\mathrm{a}_1\mathrm{b}_1\mathrm{d}_1\mathrm{c}_2\mathrm{e}_2\mathrm{f}_2}\}}\,.$$

or substituting in the first equation

$$\frac{\sqrt{(a-c)(b-c)}}{\sqrt{(a-d)(b-d)}}\cdot\frac{\sqrt{\mathrm{a}_2\mathrm{b}_2\mathrm{c}_2\mathrm{d}_1\mathrm{e}_1\mathrm{f}_1}-\sqrt{\mathrm{a}_1\mathrm{b}_1\mathrm{c}_1\mathrm{d}_2\mathrm{e}_2\mathrm{f}_2}}{\sqrt{\mathrm{a}_2\mathrm{b}_2\mathrm{d}_2\mathrm{c}_1\mathrm{e}_1\mathrm{f}_1}-\sqrt{\mathrm{a}_1\mathrm{b}_1\mathrm{d}_1\mathrm{c}_2\mathrm{e}_2\mathrm{f}_2}}=\frac{\sqrt{\mathrm{c}_3\mathrm{c}_4}}{\sqrt{\mathrm{d}_3\mathrm{d}_4}}\,.$$

159. Considering the duad DE as an abbreviation for the double triad $ABC.DEF$, the expressed duad being always accompanied by the letter F, we are thus led to the consideration of the double-letter functions

$$AB_{12}=\frac{1}{x_1-x_2}\,\{\sqrt{\mathrm{a}_1\mathrm{b}_1\mathrm{f}_1\mathrm{c}_2\mathrm{d}_2\mathrm{e}_2}-\sqrt{\mathrm{a}_2\mathrm{b}_2\mathrm{f}_2\mathrm{c}_1\mathrm{d}_1\mathrm{e}_1}\},\ \text{etc.}$$

in connection with the already mentioned single-letter functions $A_{12}=\sqrt{\mathrm{a}_1\mathrm{a}_2}$, etc. viz. in this notation the equation just obtained is

$$\frac{C_{34}}{D_{34}}=\sqrt{\frac{(a-c)(b-c)}{(a-d)(b-d)}}\,\frac{DE_{12}}{CE_{12}},$$

and it thus appears that the points 3, 4 being obtained as above from the given points 1, 2, then that the quotient of two of the single-letter functions of 3, 4 is a constant multiple of the quotient of two of the double-letter functions of 1, 2. Observe that the points 3, 4 are derived from 1, 2 by means of the two

points a, b: we have DE standing for $ABC.DEF$, CE for $ABD.CEF$, and if the two functions were represented by ABC, ABD respectively, then the form would have been
$$\frac{C_{34}}{D_{34}}=\sqrt{\frac{(a-c)(b-c)}{(a-d)(b-d)}}\cdot\frac{ABC_{12}}{ABD_{12}},$$
which is a clearer expression of the theorem; the apparent want of symmetry of the first form arises only from the arbitrary selection of the letter F to accompany the expressed duad, and is at once removed by substituting for a duad DE the triad $ABC.DEF$ which is thereby signified. The denominator factor x_1-x_2 is introduced in order to make the degree in x_1 or x_2 equal to that of the single-letter functions.

The Addition Theory. Art. Nos. 160 to 163.

160. We have the six single-letter symbols A, B, C, D, E, F; viz. $A_{12}=\sqrt{\mathrm{a}_1\mathrm{a}_2}$, etc.: and the ten double-letter symbols AB, AC, AD, AE, BC, BD, BE, CD, CE, DE, viz.
$$AB_{12}=\frac{1}{x_1-x_2}\{\sqrt{\mathrm{a}_1\mathrm{b}_1\mathrm{f}_1\mathrm{c}_2\mathrm{d}_2\mathrm{e}_2}-\sqrt{\mathrm{a}_2\mathrm{b}_2\mathrm{f}_2\mathrm{c}_1\mathrm{d}_1\mathrm{e}_1}\}, \text{ etc.}$$
these 16 functions being connected by algebraical relations which are immediately deducible from these expressions of the functions in terms of x_1, x_2. The problem is to express the functions of 5, 6 in terms of those of 1, 2 and of those of 3, 4. The relation between the variables x_1, x_2, x_3, x_4, x_5, x_6 consists herein that we have x_1, x_2, x_3, x_4, x_5, x_6 as the roots of the equation
$$[(1,x)^3]^2\alpha x^3+\beta x^2+\gamma x+\delta-\lambda(a-x)(b-x)(c-x)(d-x)(e-x)(f-x)=0;$$
or what is the same thing, it consists in the identity
$$\begin{aligned}&[\alpha x^3+\beta x^2+\gamma x+\delta]^2-\lambda(a-x)(b-x)(c-x)(d-x)(e-x)(f-x)\\&\qquad-\mu(x_1-x)(x_2-x)(x_3-x)(x_4-x)(x_5-x)(x_6-x)=0;\end{aligned}$$
or again it may be expressed by the plexus of equations
$$\left\|\begin{array}{cccccc}1, & 1, & 1, & 1, & 1, & 1\\ x_1, & x_2, & x_3, & x_4, & x_5, & x_6\\ x_1^2, & x_2^2, & x_3^2, & x_4^2, & x_5^2, & x_6^2\\ x_1^3, & x_2^3, & x_3^3, & x_4^3, & x_5^3, & x_6^3\\ \sqrt{X_1}, & \sqrt{X_2}, & \sqrt{X_3}, & \sqrt{X_4}, & \sqrt{X_5}, & \sqrt{X_6}\end{array}\right\|=0,$$
(where $X_1=(a-x_1)(b-x_1)\ldots(f-x_1)$, etc.) equivalent of course to two equations, and serving to determine x_5, x_6 in terms of x_1, x_2, x_3, x_4.

161. The solution is in fact as is given in my paper "On the Addition of the double ϑ-Functions," *Crelle*, t. 88 (1880), pp. 74–81. Writing successively

$x = x_1, x_2, x_3, x_4$, we have

$$\begin{aligned}
\alpha x_1^3 + \beta x_1^2 + \gamma x_1 + \delta &= \sqrt{\lambda}\sqrt{X_1},\\
\alpha x_2^3 + \beta x_2^2 + \gamma x_2 + \delta &= \sqrt{\lambda}\sqrt{X_2},\\
\alpha x_3^3 + \beta x_3^2 + \gamma x_3 + \delta &= \sqrt{\lambda}\sqrt{X_3},\\
\alpha x_4^3 + \beta x_4^2 + \gamma x_4 + \delta &= \sqrt{\lambda}\sqrt{X_4},
\end{aligned}$$

which equations serve to determine in terms of x_1, x_2, x_3, x_4 the ratios $\alpha:\beta:\gamma:\delta$: and we have then the two like equations

$$\begin{aligned}
\alpha x_5^3 + \beta x_5^2 + \gamma x_5 + \delta &= \sqrt{\lambda}\sqrt{X_5},\\
\alpha x_6^3 + \beta x_6^2 + \gamma x_6 + \delta &= \sqrt{\lambda}\sqrt{X_6},
\end{aligned}$$

which determine the symmetric functions of x_5, x_6.

If reverting to the identity, we write therein for instance $x = a$, we find

$$\alpha a^3 + \beta a^2 + \gamma a + \delta = \sqrt{\mu}\, A_{12} A_{34} A_{56},$$

which equation when properly reduced gives the proportional value of A_{56}.

162. Calling for a moment the function on the left-hand side Ω, we have

$$\begin{vmatrix}
x_1^3, & x_1^2, & x_1, & 1, & \sqrt{\lambda}\sqrt{X_1}\\
x_2^3, & x_2^2, & x_2, & 1, & \sqrt{\lambda}\sqrt{X_2}\\
x_3^3, & x_3^2, & x_3, & 1, & \sqrt{\lambda}\sqrt{X_3}\\
x_4^3, & x_4^2, & x_4, & 1, & \sqrt{\lambda}\sqrt{X_4}\\
a^3, & a^2, & a\ , & 1, & \Omega
\end{vmatrix} = 0$$

that is $$\Omega \begin{vmatrix}
x_1^3, & x_1^2, & x_1, & 1\\
x_2^3, & x_2^2, & x_2, & 1\\
x_3^3, & x_3^2, & x_3, & 1\\
x_4^3, & x_4^2, & x_4, & 1
\end{vmatrix} + \sqrt{\lambda} \begin{vmatrix}
x_1^3, & x_1^2, & x_1, & 1, & \sqrt{X_1}\\
x_2^3, & x_2^2, & x_2, & 1, & \sqrt{X_2}\\
x_3^3, & x_3^2, & x_3, & 1, & \sqrt{X_3}\\
x_4^3, & x_4^2, & x_4, & 1, & \sqrt{X_4}
\end{vmatrix} = 0,$$

viz. this is

$$\begin{aligned}
&\Omega\,(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4)\\
&\quad = -\sqrt{\lambda}\,\{\sqrt{X_1}\,.\,x_2 - x_3\,.\,x_2 - x_4\,.\,x_2 - a\,.\,x_3 - x_4\,.\,x^3 - a\,.\,x_4 - a\\
&\qquad + \sqrt{X_2}\,.\,x_3 - x_4\,.\,x_3 - a\,.\,x_3 - x_1\,.\,x_4 - a\,.\,x_4 - x_1\,.\,a - x_1\\
&\qquad + \sqrt{X_3}\,.\,x_4 - a\,.\,x_4 - x_1\,.\,x_4 - x_2\,.\,a - x_1\,.\,a - x_2\,.\,x_1 - x_2\\
&\qquad + \sqrt{X_4}\,.\,a - x_1\,.\,a - x_2\,.\,a - x_3\,.\,x_2 - x_3\,.\,x_2 - x_4\,.\,x_3 - x_4\},
\end{aligned}$$

or as this may be written

$$\begin{aligned}
&\Omega\,.\,x_1 - x_3\,.\,x_1 - x_4\,.\,x_2 - x_3\,.\,x_2 - x_4\\
&= \frac{\sqrt{\lambda}\,.\,a - x_3\,.\,a - x_4}{x_1 - x_2}\,\{x_2 - x_3\,.\,x_2 - x_4\,.\,a - x_2\,.\,\sqrt{X_1} - .\,x_1 - x_3\,.\,x_1 - x_4\,.\,a - x_1\,.\,\sqrt{X_2}\}\\
&+ \frac{\sqrt{\lambda}\,.\,a - x_1\,.\,a - x_2}{x_3 - x_4}\,\{x_4 - x_1\,.\,x_4 - x_2\,.\,a - x_4\,.\,\sqrt{X_3} - .\,x_3 - x_1\,.\,x_3 - x_2\,.\,a - x_3\,.\,\sqrt{X}\},
\end{aligned}$$

we have here $\sqrt{\lambda}\,.\,a - x_3\,.\,a - x_4 = \sqrt{\lambda}\,A_{34}^2$, and the function

$$\frac{1}{x_1 - x_2}\,\{x_2 - x_3\,.\,x_2 - x_4\,.\,\mathrm{a}_2\sqrt{X_1} - .\,x_1 - x_3\,.\,x_1 - x_4\,.\,\mathrm{a}_1\sqrt{X_2}\},$$

which multiplies this is without difficulty found to be

$$= \frac{-A_{12}}{c-d.d-b.b-c} \Sigma\{c-d.B_{34}^2 BE_{12}.C_{12}.D_{12}\},$$

where the summation extends to the three terms obtained by the cyclical interchange of the letters b, c, d: these being a set of three out of the five letters other than a. Similarly $\sqrt{\lambda}.a-x_1.a-x_2$ is $=\sqrt{\lambda}A_{12}^2$, and the function which multiplies this is

$$= \frac{-A_{34}}{c-d.d-b.b-c} \Sigma\{c-d.B_{12}^2.BE_{34}C_{14}D_{14}\},$$

the expression for Ω thus contains the factor $A_{12}A_{34}$. But we have

$$\Omega, = a\alpha^3 + b\alpha^2 + c\alpha + d, = \sqrt{\mu}A_{12}A_{34}A_{56};$$

this equation contains therefore the factor $A_{12}A_{34}$, and omitting it we find

$$-\frac{\sqrt{\mu}}{\sqrt{\lambda}}(x_1-x_3.x_1-x_4.x_2-x_3.x_2-x_4)(c-d.d-b.b-c)A_{56}$$
$$= A_{34}\Sigma\{c-d.B_{34}^2 BE_{12}C_{12}D_{12}\} + A_{12}\Sigma\{c-d.B_{12}^2 BE_{34}C_{34}D_{14}\},$$

where as before the summations refer each to the three terms obtained by the cyclical interchange of the letters b, c, d; these being any three of the five letters other than a: and the remaining two letters e, f enter into the formulæ symmetrically. The formula thus gives for A_{56} ten values which are of course equal to each other.

By reason of the undetermined factor $\frac{\sqrt{\mu}}{\sqrt{\lambda}}$ the formula gives only the proportional value of A_{56}; viz. combining it with the like formulæ for B_{56}, etc. we have determinate values of the ratios $A_{56} : B_{56} \ldots : F_{56}$. But this being understood, we regard the formula as a formula for each single-letter function of x_5, x_6 in terms of the single and double-letter functions of x_1, x_2 and of x_3, x_4 respectively

163. We require further the expressions for the double-letter functions of x_5, x_6. Consider for example the function DE_{56} which is

$$= \frac{1}{x_5-x_6}\{\sqrt{\mathrm{d}_5\mathrm{e}_5\mathrm{f}_5\mathrm{a}_6\mathrm{b}_6\mathrm{c}_6} - \sqrt{\mathrm{d}_6\mathrm{e}_6\mathrm{f}_6\mathrm{a}_5\mathrm{b}_5\mathrm{c}_5}\}$$

then multiplying by $A_{56}B_{56}C_{56}, = \sqrt{\mathrm{a}_5\mathrm{b}_5\mathrm{c}_5\mathrm{a}_6\mathrm{b}_6\mathrm{c}_6}$, we have

$$DE_{56}A_{56}B_{56}C_{56} = \frac{1}{x_5-x_6}\{\mathrm{a}_6\mathrm{b}_6\mathrm{c}_6\sqrt{X_5} - \mathrm{a}_5\mathrm{b}_5\mathrm{c}_5\sqrt{X_6}\},$$

or recollecting that $\sqrt{\lambda}\sqrt{X_5}$ and $\sqrt{\lambda}\sqrt{X_6}$ are $=\alpha x_5^3+\beta x_5^2+\gamma x_5+\delta$, and $\alpha x_6^3+\beta x_6^2+\gamma x_6+\delta$ respectively, this may be written

$$\sqrt{\lambda}\,DE_{56}A_{56}B_{56}C_{56} = \frac{1}{x_5-x_6}\{a-x_6.b-x_6.c-x_6.(\alpha x_5^3+\beta x_5^2+\gamma x_5+\delta)$$
$$-.a-x_5.b-x_5.c-x_5.(\alpha x_6^3+\beta x_6^2+\gamma x_6+\delta)\}.$$

Using the well-known identity

$$\alpha x_5^3 + \beta x_5^2 + \gamma x_5 + \delta = \Sigma . \alpha a^3 + \beta a^2 + \gamma a + \delta . \frac{b - x_5 . c - x_5 . d - x_5}{b - a . c - a . d - a},$$

where the summation extends to the four terms obtained by the cyclical interchanges of the letters a, b, c, d: and the like identity for $\alpha x_6^3 + \beta x_6^2 + \gamma x_6 + \delta$, there will be terms in $\alpha a^3 + \beta a^2 + \gamma a + \delta$, $\alpha b^3 + \beta b^2 + \gamma b + \delta$, $\alpha c^3 + \beta c^2 + \gamma c + \delta$, but the term in $\alpha d^3 + \beta d^2 + \gamma d + \delta$ will disappear of itself. After some easy reductions the result is

$$\surd\lambda \, DE_{56} A_{56} B_{56} C_{56} = \Sigma \frac{\alpha a^3 + \beta a^2 + \gamma a + \delta}{b - a . c - a} B_{56}^2 C_{56}^2,$$

where the summation extends to the three terms obtained by the cyclical interchanges of the letters a, b, c. We have $\alpha a^3 + \beta a^2 + \gamma a + \delta = \surd\mu . A_{12} A_{34} A_{56}$, and similarly for the other two terms; the whole equation thus divides by $A_{56} B_{56} C_{56}$, and we find

$$-\frac{\surd\mu}{\surd\lambda} DE_{56} = \frac{1}{b - c . c - a . a - b} \left(\frac{\surd\mu}{\surd\lambda}\right)^2 . \Sigma (b - c . A_{12} A_{34} B_{56} C_{56},$$

in which equation, if we imagine $\frac{\surd\mu}{\surd\lambda} A_{56}$, $\frac{\surd\mu}{\surd\lambda} B_{56}$, $\frac{\surd\mu}{\surd\lambda} C_{56}$, each replaced by its value in terms of the single and double-letter functions of x_1, x_2 and x_3, x_4, we have an equation of the form

$$-\frac{\surd\mu}{\surd\lambda} (x_1 - x_3 . x_1 - x_4 . x_2 - x_3 . x_2 - x_4) DE_{56} = \frac{1}{x_1 - x_3 . x_1 - x_4 . x_2 - x_3 . x_2 - x_4} M,$$

where M is a given rational and integral function of the single and double-letter functions of x_1, x_2 and x_3, x_4. The factor on the left-hand side has been made the same as in the formula for the single-letter functions A_{56}, etc., and to do this it was necessary to bring in on the right-hand side the factor

$$\frac{1}{x_1 - x_3 . x_1 - x_4 . x_2 - x_3 . x_2 - x_4};$$

this disappears in the expression for the ratio of two double-letter functions; but it enters into the expression for the ratio of a single-letter to a double-letter function, and it then requires to be itself expressed in terms of the functions of x_1, x_2 and x_3, x_4: it is easy to see that we have

$$x_1 - x_3 . x_1 - x_4 . x_2 - x_3 . x_2 - x_4 = \Sigma \frac{(A_{34}^2 B_{12}^2 - A_{12}^2 B_{34}^2)(A_{34}^2 C_{12}^2 - A_{12}^2 C_{34}^2)}{(a - b)^2 (a - c)^2},$$

where the summation extends to the three terms obtained by the cyclical interchanges of the letters a, b, c: these being a set of any three out of the six letters.

We have, in what precedes, obtained the expressions for the ratios of the 16 functions $A_{56}, \ldots F_{56}, AB_{56}, \ldots DE_{56}$ in terms of the ratios of the like functions of x_1, x_2 and x_3, x_4.

CHAPTER VII. THE FUNCTIONS T, U, V, Θ.

The present chapter is substantially a reproduction of C. & G.'s seventh section, "Die Function $T_{\xi\eta}$" (borrowing only from the next section the definition of the theta-function), but for greater simplicity I consider for the most part, the case, fixed curve a quartic; $n = 4$, $p = 3$.

Integral Form of the Affected Theorem. Art. Nos. 164 to 169.

164. Writing for shortness $\dfrac{(x, y, z)_{12}^{n-2}\, d\omega}{012} = d\Pi_{12}$, we are concerned with the integrals $\int_{a'}^{a} d\Pi_{12}$ which present themselves in connection with the affected theorem: the notation is explained, Chap. V; a, a' are points on the curve f; the variable may be any parameter serving for the determination of the current point, and the integral, taken from the value which belongs to the point a' to the value which belongs to the point a, is represented as above by means of the two points a, a' as limits of the integral. It is assumed that the integral is a canonical integral having the limits and the parametric points interchangeable, $\int_{a'}^{a} d\Pi_{12} = \int_{2}^{1} d\Pi_{aa'}$: see Chapter IV.

165. Writing for shortness

$$\left(\int_{a'}^{a} + \int_{b'}^{b} + \int_{c'}^{c} + \ldots\right) d\Pi_{12} = \int \begin{pmatrix} a, & b, & c & \ldots \\ a', & b', & c' & \ldots \end{pmatrix} d\Pi_{12}$$

then if ϕ, ψ are curves each of the order m, the former of them intersecting the fixed curve f in the points $a, b, c \ldots$, and the latter of them intersecting the same curve in the points $a', b', c' \ldots$, and if ϕ_1, ψ_1, ϕ_2, ψ_2 are what the functions ϕ, ψ become on substituting therein in place of the current coordinates the values which belong to the parametric points 1, 2 respectively; the theorem becomes

$$\int \begin{pmatrix} a, & b, & c & \ldots \\ a', & b', & c' & \ldots \end{pmatrix} d\Pi_{12} = \log \frac{\varphi_2 \psi_1}{\varphi_1 \psi_2}.$$

The superior limits may be interchanged in any manner, and so also the inferior limits may be interchanged in any manner. If a superior limit coincide with an inferior limit, the two may thus be considered as belonging to an integral which will then have the value 0, and the coincident points may therefore be omitted from the expression on the left-hand side: and so in the case of any number of coincidences.

166. If the intersections of the curves ϕ, ψ and the parametric points are situate on a curve of the order m; then taking the equation of this curve to be $\phi + \lambda\psi = 0$, we have simultaneously $\phi_1 + \lambda\psi_1 = 0$, $\phi_2 + \lambda\psi_2 = 0$; whence $\phi_2\psi_1 = \psi_1\phi_2$, and the logarithmic term disappears: viz. the theorem becomes

$$\int\begin{pmatrix} a\,, b\,, c\,\ldots \\ a', b', c'\ldots \end{pmatrix} d\Pi_{12} = 0\,.$$

167. Suppose that the curves ϕ, ψ are each of them a major curve, that is a curve of the order $n - 2$ passing through the δ dps, and consequently besides meeting the curve f in $n(n-2) - 2\delta$, $= 2p + n - 2$ points: the theorem is

$$\int\begin{pmatrix} a\,, b\,, c\,\ldots \\ a', b', c'\ldots \end{pmatrix} d\Pi_{12} = \log\frac{\varphi_2\psi_1}{\varphi_1\psi_2},$$

where the numbers of the superior and of the inferior points are each $= 2p + n - 2$.

168. Suppose further that the curves ϕ, ψ, being major curves as above, pass each of them through the $n - 2$ residues of $1, 2$; they besides meet in $(n-2)(n-3)$ points (viz. these are the δ dps and $(n-2)(n-3) - \delta$ variable points): these $(n-2)(n-3)$ points lie on a minor curve, that is a curve of the order $n - 3$ passing through the dps; and the minor curve together with the parametric line 12 make together a major curve passing through the intersections of ϕ, ψ and also through the parametric points $1, 2$: viz. these points and the intersections of ϕ, ψ are situate on a curve of the order $n - 2$; the logarithmic term thus vanishes. The intersections of ϕ with the fixed curve are the δ dps, the $n - 2$ residues and $2p$ other points, say these are $a, b, c\ldots, a^{\times}, b^{\times}, c^{\times}, \ldots$; similarly the curve ψ meets the fixed curve in the δ dps, the $n - 2$ residues, and in $2p$ other points, say these are $d, e, f, \ldots, d^{\times}; e^{\times}, f^{\times}, \ldots$: the theorem is

$$\int\begin{pmatrix} a, b, c\ldots a^{\times}, b^{\times}, c^{\times}\ldots \\ d, e, f\ldots d^{\times}, e^{\times}, f^{\times}\ldots \end{pmatrix} d\Pi_{12} = 0,$$

where there are $2p$ superior and inferior points respectively.

169. I introduce the definitions: a minor curve meets the fixed curve in the dps and in $2p - 2$ other points, called "cominors": a major curve passing through the $n - 2$ residues of the points $1, 2$, meets the fixed curve in the δ dps, the $n - 2$ residues and in $2p$ other points, called "comajors in regard to the points $1, 2$". Observe that $p - 1$ of the cominors determine uniquely the remaining $p - 1$ cominors; and similarly p of the comajors determine uniquely the remaining p comajors.

The foregoing theorem thus is that the sum $\int (\ ) d\Pi_{12}$ is $= 0$, when the superior points and the inferior points are each of them a system of comajors in regard to the parametric points 1, 2.

Fixed Curve a Quartic. Art. No. 170.

170. It would be easy to go on with the general form, but as already mentioned, I prefer to consider the case, fixed curve a quartic; $n = 4$, $p = 3$. A minor curve is here a line meeting the quartic in 4 points, which are "cominors"; the major curve is a conic, and if this passes through the residues of 1, 2 it besides meets the quartic in 6 points, which are "comajors in regard to the points 1, 2". Two points and their residues are cominors, but this is only by reason that $n - 3 = 1$.

The Function T. Art. No. 171.

171. In conformity with C. & G. I introduce the functional symbol

$$T_{12}\begin{pmatrix} a, & b, & c \dots \\ a', & b', & c' \dots \end{pmatrix} = \int \begin{pmatrix} a, & b, & c \dots \\ a', & b', & c' \dots \end{pmatrix} d\Pi_{12},$$

so that T denotes a function of the parametric points, and of the sets of superior and inferior points respectively. The foregoing theorem for the quartic thus is

$$T_{12}\begin{pmatrix} a, & b, & c, & a^{\times}, & b^{\times}, & c^{\times} \\ d, & e, & f, & d^{\times}, & e^{\times}, & f^{\times} \end{pmatrix} = 0.$$

Observing that $\int_d^a + \int_{d^{\times}}^{a^{\times}} = 2\int_d^a - \int_{a^{\times}}^a + \int_{d^{\times}}^d$, and so in other cases, this may be written

$$2T_{12}\begin{pmatrix} a, & b, & c \\ d, & e, & f \end{pmatrix} = T_{12}\begin{pmatrix} a, & b, & c \\ a^{\times}, & b^{\times}, & c^{\times} \end{pmatrix} - T_{12}\begin{pmatrix} d, & e, & f \\ d^{\times}, & e^{\times}, & f^{\times} \end{pmatrix},$$

and if as a definition of $T_{12}(a, b, c)$, we write

$$T_{12}(a, b, c) = T_{12}\begin{pmatrix} a, & b, & c \\ a^{\times}, & b^{\times}, & c^{\times} \end{pmatrix},$$

where $a^{\times}$, $b^{\times}$, $c^{\times}$ are the comajors of a, b, c in regard to 1, 2, then the equation is

$$2T_{12}\begin{pmatrix} a, & b, & c \\ d, & e, & f \end{pmatrix} = T_{12}(a, b, c) - T_{12}(d, e, f),$$

viz. the function of the $(2p + 2 =)8$ points $1, 2, a, b, c, d, e, f$ is here expressed as a difference of two functions each of $(p + 2 =)5$ points: $T_{12}(a, b, c)$ is regarded as a function of the 5 points $1, 2, a, b, c$, because the remaining points $a^{\times}$, $b^{\times}$, $c^{\times}$ depend only on these 5 points.

The Function U. Art. Nos. 172 to 175.

172. We consider on the quartic the points ξ, μ; 1, 2, 3; and taking f, f' for the cominors of 2, 3; g, g' for the cominors of 3, 1; and h, h' for the

cominors of 1, 2, we write

$$T = T_{\xi\mu}(1, 2, 3),$$

$$T_1 = T_{1\mu}(\xi, f, f'),\ T_2 = T_{2\mu}(\xi, g, g'),\ T_3 = T_{3\mu}(\xi, h, h');$$

it is to be shown that there exists a function $U(1, 2, 3;\, \xi)$ such that

$$\delta U = \tfrac{1}{2}\{\delta_\xi T + \delta_1 T_1 + \delta_2 T_2 + \delta_3 T_3\},$$

viz. considering ξ, 1, 2, 3 as variable points on the quartic, the whole infinitesimal variation of U is the sum of these parts, where $\delta_\xi T$ is the variation of T when only ξ is varied, $\delta_1 T_1$ the variation of T_1 when only 1 is varied, and similarly for $\delta_2 T_2$ and $\delta_3 T_3$. We consider in the proof three other points 4, 5, 6 on the quartic; and taking l, l' for the cominors of 5, 6; m, m' for those of 6, 4; and n, n' for those of 4, 5, we write further

$$X_1 = T_{1\mu}(4, l, l'),\ X_2 = T_{2\mu}(5, m, m'),\ X_3 = T_{3\mu}(6, n, n'),$$

and it then requires to be shown that

$$\begin{aligned}
\tfrac{1}{2}T \qquad &= \int \begin{pmatrix} 123 \\ 456 \end{pmatrix} d\Pi_{\xi\mu} + \tfrac{1}{2} T_{\xi\mu}(4, 5, 6),\\
\tfrac{1}{2}(T_1 - X_1) &= \int \begin{pmatrix} \xi 56 \\ 423 \end{pmatrix} d\Pi_{1\mu} + \log \frac{\mu 23.156}{\mu 56.123},\\
\tfrac{1}{2}(T_2 - X_2) &= \int \begin{pmatrix} \xi 64 \\ 531 \end{pmatrix} d\Pi_{2\mu} + \log \frac{\mu 31.264}{\mu 64.123},\\
\tfrac{1}{2}(T_3 - X_3) &= \int \begin{pmatrix} \xi 45 \\ 612 \end{pmatrix} d\Pi_{3\mu} + \log \frac{\mu 12.345}{\mu 45.123},
\end{aligned}$$

where $\mu 23$ is the determinant formed with the coordinates of the points μ, 2, 3 respectively: and so in other cases.

173. We have $T_{\xi\mu}\begin{pmatrix} 1, & 2, & 3 \\ 4, & 5, & 6 \end{pmatrix} = \tfrac{1}{2} T_{\xi\mu}(1, 2, 3) - \tfrac{1}{2} T_{\xi\mu}(4, 5, 6),$

that is $\qquad = \tfrac{1}{2} T \qquad - \tfrac{1}{2} T_{\xi\mu}(4, 5, 6),$

and thence the above value of $\tfrac{1}{2}T$.

The affected theorem gives

$$\int \begin{pmatrix} f, & f', & 2, & 3 \\ l, & l', & 5, & 6 \end{pmatrix} d\Pi_{1\mu} = \log \frac{F_\mu L_1}{F_1 L_\mu},$$

where $F = 0$ is the equation of the line through f, f', 2, 3; and F_1, F_μ are what the function F becomes on substituting therein for the current coordinates the coordinates of the points 1, μ respectively. And similarly $L = 0$ is the equation of the line through l, l', 5, 6; and L_1, L_μ are what the function L becomes by the same substitutions respectively. The values of F_1, F_2 are 123, $\mu 23$: those of L_1, L_μ are 156, $\mu 56$, and the logarithmic term is thus

$$= \log \frac{\mu 23.156}{\mu 56.123}.$$

We then have

$$\tfrac{1}{2}(T_1-X_1)=\int\begin{pmatrix}\xi, & f, & f' \\ 4, & l, & l'\end{pmatrix}d\Pi_{1\mu},\ =\int\begin{pmatrix}\xi, & 5, & 6 \\ 4, & 2, & 3\end{pmatrix}d\Pi_{1\mu}+\int\begin{pmatrix}f, & f', & 2, & 3 \\ l, & l', & 5, & 6\end{pmatrix}d\Pi_{1\mu},$$

and in this last expression for $\tfrac{1}{2}(T_1-X_1)$ substituting for the second term the logarithmic value just obtained we have the required expression for $\tfrac{1}{2}(T_1-X_1)$: and those for $\tfrac{1}{2}(T_2-X_2)$ and $\tfrac{1}{2}(T_3-X_3)$, are deduced by mere cyclical permutations of the letters.

174. Returning to the assumed relation $\delta U=\tfrac{1}{2}\{\delta_\xi T+\delta_1 T_1+\delta_2 T_2+\delta_3 T_3\}$; in order to the existence of the function U, it is only necessary to show that $T-T_1$ contains no term in $1, \xi$ (that is no term depending on both these points), and that T_1-T_2 contains no term in $1, 2$: for then by symmetry the like properties hold in regard to $T-T_2$, $T-T_3$, T_1-T_3, T_2-T_3 respectively, and the assumed expression is a complete differential, from which the function U may be obtained by integration.

175. To show that $T-T_1$ contains no term in $1, \xi$.

$$\text{For } T, \text{ the only term in } 1, \xi \text{ is } \int_4^1 d\Pi_{\xi\mu},$$

$$\text{,, } T_1 \quad\text{,,}\quad\text{,,}\quad \int_4^\xi d\Pi_{1\mu},$$

and it is to be shown that the difference of the two integrals contains no term in $1, \xi$. Considering on the quartic the two new points δ, ε, the first integral is

$$\int\begin{pmatrix}1, & \delta \\ \delta, & 4\end{pmatrix}(d\Pi_{\xi\epsilon}+d\Pi_{\epsilon\mu}),\ =\int_\delta^1 d\Pi_{\xi\epsilon}+\int_4^\delta d\Pi_{\xi\epsilon}+\int_4^1 d\Pi_{\epsilon\mu},$$

and the second is

$$\int\begin{pmatrix}\xi, & \varepsilon \\ \varepsilon, & 4\end{pmatrix}(d\Pi_{1\delta}+d\Pi_{\delta\mu}),\ =\int_\epsilon^\xi d\Pi_{1\delta}+\int_4^\epsilon d\Pi_{\delta\mu}+\int_\mu^\xi d\Pi_{\delta\mu}.$$

Hence in the difference the only terms which can contain $1, \xi$ is

$$\int_\delta^1 d\Pi_{\xi\epsilon}-\int_\epsilon d\Pi_{1\delta}$$

and this term is $=0$: wherefore there is not in the difference any term in $1, \xi$. This proves the property for $T-T_1$. The property for T_1-T_2 is proved in a similar manner.

Theorems in regard to the Function U. Art. Nos. 176 to 179.

176. Theorem (A).

$$U(1,\ 2,\ 3;\ \xi)-U(1,\ 2,\ 3;\ \mu)=\tfrac{1}{2}\,T_{\xi\mu}(1,\ 2,\ 3), \qquad \text{(A)}$$

we have

$$U(1,\ 2,\ 3;\ \xi)-U(1,\ 2,\ 3;\ \mu)=\int_\mu^\xi d_\xi U=\tfrac{1}{2}\int_\mu^\xi d_\xi T$$

$$=\tfrac{1}{2}T_{\xi\mu}(1,\ 2,\ 3)-\tfrac{1}{2}T_{\mu\mu}(1,\ 2,\ 3),$$

and $T_{\mu\mu}(1,\ 2,\ 3) = \int \begin{pmatrix} 1, & 2, & 3 \\ 1^{\times} & 2^{\times} & 3^{\times} \end{pmatrix} d\Pi_{\mu\mu}$, where $d\Pi_{\mu\mu} = 0$, viz. considering this as derived from $d\Pi_{\xi\mu},\ = Q_{\xi\mu}\, d\omega$, by making the point ξ coincide with μ, then when ξ is indefinitely near to μ, the numerator and denominator of $Q_{\xi\mu}$ are each of them infinitesimal of the orders 3 and 2 respectively, and thus the function $Q_{\xi\mu}$ ultimately vanishes (see as to this, Chap. V. Art. Nos. 99 to 106). We have therefore $T_{\mu\mu}(1,\ 2,\ 3) = 0$, and the required theorem is proved.

177. Theorem (B).

$$U(1,\ 2,\ 3\ ;\ \xi) - U(4,\ 2,\ 3\ ;\ \xi) = \tfrac{1}{2}\, T_{14}(\xi, f, f'), \qquad \text{(B)}$$

where as before f, f' are the cominors of 2, 3, that is 2, 3, f, f' lie on a line. We have

$$U(1,\ 2,\ 3\ ;\ \xi) - U(4,\ 2,\ 3\ ;\ \xi) = \int_4^1 d_1 U = \tfrac{1}{2}\int_4^1 d_1 T_1$$

$$= \tfrac{1}{2}\, T_{1\mu}(\xi, f, f') - \tfrac{1}{2}\, T_{4\mu}(\xi, f, f'),$$

the point μ is arbitrary, and it may be taken to coincide with 4; but we then have $T_{44}(\xi, f, f') = 0$, and the theorem is thus proved.

178. Theorem (C). $T_{11^{\times}}(\xi, f, f') + T_{11^{\times}}(\eta,\ k,\ k') = 0$: (C)

where ξ, η, 1, 2, 3 are arbitrary points on the quartic; $1^{\times}$, $2^{\times}$, $3^{\times}$ are the comajors of 1, 2, 3 in regard to ξ, η, viz. the points 1, 2, 3, $1^{\times}$, $2^{\times}$, $3^{\times}$ lie on a conic which passes through ξ', η' the residues (or cominors) of ξ, η : f, f' are the cominors of 2, 3; and k, k' are the cominors of $2^{\times}$, $3^{\times}$.

Taking θ, θ' for the cominors of 1, $1^{\times}$, the four lines $\xi\eta\xi'\eta'$, $11^{\times}\theta\theta'$, $23ff'$ and $2^{\times}3^{\times}kk'$ form a quartic cutting the fixed quartic in the 16 points: but of these ξ', η', 1, 2, 3, $1^{\times}$, $2^{\times}$, $3^{\times}$ lie in a conic: hence the remaining 8 points θ, θ', ξ, η, f, f', k, k' lie in a conic; that is, ξ, η, f, f', k, k' lie on a conic through θ, θ', the residues of 1, $1^{\times}$, or they are comajors in regard to 1, $1^{\times}$; whence the theorem.

179. We have

From A. From B.

$$\tfrac{1}{2}\, T_{11^{\times}}(\xi, f, f') = U(\xi, f, f'; 1) - U(\xi, f, f'; 1^{\times}),\ = U(1,\ 2,\ 3;\ \xi) - U(1^{\times},\ 2,\ 3;\ \xi),$$

$$\tfrac{1}{2}\, T_{11^{\times}}(\eta,\ k,\ k') = U(\eta,\ k,\ k';\ 1) - U(\eta,\ k,\ k';\ 1^{\times}),\ = U(1,\ 2^{\times},\ 3^{\times};\ \eta) - U(1^{\times},\ 2,\ 3;\ \eta),$$

viz. we have thus two expressions for each term of the equation (C),

$$T_{11^{\times}}(\xi, f, f') + T_{11^{\times}}(\eta,\ k,\ k') = 0.$$

In particular we have Theorem (D)

$$U(1,\ 2,\ 3\ ;\ \xi) - U(1^{\times},\ 2,\ 3\ ;\ \xi) = -\,U(1,\ 2^{\times},\ 3^{\times};\ \xi) + U(1^{\times},\ 2^{\times},\ 3^{\times};\ \eta). \qquad \text{(D)}$$

Again we have $T_{\xi\eta}(1,\ 2,\ 3) + T_{\xi\eta}(1^{\times},\ 2^{\times},\ 3^{\times}) = 0$; where 1, 2, 3, $1^{\times}$, $2^{\times}$, $3^{\times}$ are comajors in regard to ξ, η: and

$$\tfrac{1}{2}\, T_{\xi\eta}(1,\ 2,\ 3) = U(1,\ 2,\ 3\ ;\ \xi) - U(1,\ 2,\ 3\ ;\ \eta),$$

$$\tfrac{1}{2}\, T_{\xi\eta}(1^{\times},\ 2^{\times},\ 3^{\times}) = U(1^{\times},\ 2^{\times},\ 3^{\times},\ \xi) - U(1^{\times},\ 2^{\times},\ 3^{\times},\ \eta),$$

whence Theorem (E),

$$U(1,\ 2,\ 3;\ \xi)-U(1,\ 2,\ 3;\ \eta)=-U(1^\times,\ 2^\times,\ 3^\times;\ \xi)+U(1^\times,\ 2^\times,\ 3^\times,\ \eta). \qquad \text{(E)}$$

The Function V. Art. Nos. 180 to 182.

180. It is convenient to consider U as a logarithm, say $-U(1,\ 2,\ 3;\ \xi) = \log V(1,\ 2,\ 3;\ \xi)$, or $V(1,\ 2,\ 3;\ \xi)=\exp. -U(1,\ 2,\ 3;\ \xi)$. V, like U, is a function of the $(p+1=)4$ points $1,\ 2,\ 3;\ \xi$, on the quartic.

The equation (D) thus becomes

$$\frac{V(1,\ 2,\ 3;\ \xi)}{V(1^\times,\ 2^\times,\ 3^\times;\ \eta)}=\frac{V(1^\times,\ 2,\ 3;\ \xi)}{V(1,\ 2^\times,\ 3^\times;\ \eta)},$$

where $1,\ 2,\ 3,\ 1^\times,\ 2^\times,\ 3^\times$ are comajors in regard to $\xi,\ \eta$: the equation shows that in the function $\dfrac{V(1,\ 2,\ 3;\ \xi)}{V(1^\times,\ 2^\times,\ 3^\times;\ \eta)}$ we can without alteration of the value interchange a pair of points $1,\ 1^\times$ out of the system of comajor points; and it of course follows that we can in any manner whatever interchange these points, so as to have any three of them in the numerator function and the remaining three in the denominator function. In particular we have

$$\frac{V(1,\ 2,\ 3;\ \xi)}{V(1^\times,\ 2^\times,\ 3^\times;\ \eta)}=\frac{V(1^\times,\ 2^\times,\ 3^\times;\ \xi)}{V(1,\ 2,\ 3;\ \eta)}.$$

The equation (E) becomes

$$\frac{V(1,\ 2,\ 3;\ \xi)}{V(1,\ 2,\ 3;\ \eta)}=\frac{V(1^\times,\ 2^\times,\ 3^\times;\ \eta)}{V(1^\times,\ 2^\times,\ 3^\times;\ \xi)},$$

and multiplying we find

$$V^2(1,\ 2,\ 3;\ \xi)=V^2(1^\times,\ 2^\times,\ 3^\times;\ \eta),$$

that is $V(1,\ 2,\ 3;\ \xi)=\pm V(1^\times,\ 2^\times,\ 3^\times;\ \eta)$, the sign being determinately $+$ or determinately $-$, according to the precise definition of the function V.

181. Considering η and also $1^\times,\ 2^\times,\ 3^\times$ as fixed points on the curve; but ξ as a variable point (that is, the parametric line $\xi\eta$ as rotating about the fixed point η), the points $1,\ 2,\ 3$ are then determined as the remaining intersections with the quartic, of the conic which passes through the points $1^\times,\ 2^\times,\ 3^\times$ and the points $\xi',\ \eta'$ which are the residues of $\xi,\ \eta$. And by the theorem just obtained it appears that, $\xi,\ 1,\ 2,\ 3$ thus varying, the function $V(1,\ 2,\ 3;\ \xi)$ remains constant. This comes to saying that V considered as a function of the points $1,\ 2,\ 3,\ \xi$ satisfies a certain linear partial differential equation of the first order, having a solution $V=F(u,\ v,\ w)$, an arbitrary function of $u,\ v,\ w$, determinate functions of the points $1,\ 2,\ 3,\ \xi$. And if we can find $u,\ v,\ w$ functions of these points

such that they each of them remain constant when the points 1, 2, 3, ξ vary as above, then the arbitrary function of u, v, w will remain constant for the variation in question and will thus be a value of the function V.

182. It is easily seen that such functions are

$$u,\ v,\ w = \left(\int^1 + \int^2 + \int^3 - \int^\xi\right) x\,d\omega,\ y\,d\omega,\ z\,d\omega,$$

the inferior limit being given points which are regarded as absolute constants. For by the pure theorem we have

$$\Sigma\,(x,\ y,\ z)^1 d\omega = 0,$$

where $(x, y, z)^1$ is an arbitrary linear function, and where the summation extends to all the intersections of the quartic with any given curve. Writing

$$p = \int x\,d\omega,\ \int y\,d\omega \text{ or } \int z\,d\omega, \text{ that is, } p_1 = \int^1 x\,d\omega,\ \int^1 y\,d\omega \text{ or } \int^1 z\,d\omega,$$

the inferior limits being any absolutely fixed point on the curve, and similarly p_2, etc.; the integral form of the theorem is $\Sigma p =$ constant. And applying the theorem successively to the parametric line, and to the conic which determines the points 1, 2, 3, we have

$$p_\xi + p_\eta + p_{\xi'} + p_{\eta'} = \text{const.},$$
$$p_{\xi'} + p_{\eta'} + p_1 + p_2 + p_3 + p_{1^\times} + p_{2^\times} + p_{3^\times} = \text{const.}$$

Taking the difference of these equations

$$p_1 + p_2 + p_3 - p_\xi + p_{1^\times} + p_{2^\times} + p_{3^\times} - p_\eta = \text{const.},$$

viz. the points η, $1^\times$, $2^\times$, $3^\times$ being fixed points, this is

$$p_1 + p_2 + p_3 - p_\xi = \text{const.},$$

that is, the functions u, v, w defined as above are each of them constant under the variation in question.

The Function Θ. Art. Nos. 183 and 184.

183. The function $V(1,\ 2,\ 3\ ;\ \xi)$ of the $(p + 1 =) 4$ points 1, 2, 3, ξ, is thus a function of the $(p =)$ 3 arguments

$$u,\ v,\ w, = \left(\int^1 + \int^2 + \int^3 - \int^\xi\right) x\,d\omega,\ y\,d\omega,\ z\,d\omega.$$

Disregarding a constant and exponential factor we say that it is a theta-function of these arguments, and we write the result provisionally in the form

$$V(1,\ 2,\ 3\ ;\ \xi) = \Theta(u,\ v,\ w),$$

the more precise definition of the theta-function being reserved for further consideration.

184. It appears by what precedes that a sum of ($p =$) 3 integrals $\int\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} d\Pi_{12}$, otherwise called $T_{12}\begin{pmatrix} a, & b, & c \\ d, & e, & f \end{pmatrix}$ is in the first place expressed (see No. 171) as a difference, $= \frac{1}{2} T_{12}(a, b, c) - \frac{1}{2} T_{12}(d, e, f)$ of two functions T. Each of these is by theorem (A) (No. 176) expressed as a difference of two functions U, that is as the difference of the logarithms, or logarithm of the quotient, of two functions V: such function V is according to its original definition a function of ($p + 1 =$) 4 points, but in such wise that the function is expressable as a function of ($p =$) 3 arguments, and so expressed it is a Θ-function of these arguments: and the final result thus is that the sum of ($p =$) 3 integrals $\int\begin{pmatrix} a, & b, & c \\ d, & e, & f \end{pmatrix} d\Pi_{12}$ is equal to the logarithm of a fraction, whereof the numerator and denominator are each of them a product of two Θ-functions.

END OF CHAPTER VII.